Empirical analysis: jumps in stochastic volatility model

Master's Thesis submitted

to

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Abstract

The analysis of jump detection using high frequency data has played its important role in modern financial modeling recently. In this thesis, three different algorithms, i.e. the dynamic algorithm, the highs and lows algorithm and the Gumbel test algorithm are compared for jump detection.

The dynamic algorithm, developed by Lee and Mykland [?], focuses on the determination of jump dynamics and its distribution. Simultaneously, the direction and location of jumps can be determined. It is empirically and theoretically proved that the misclassification of jumps becomes negligible using high frequency data.

The highs and lows algorithm, introduced by Kloessner [?], builds the test statistic on intradaily highest and lowest log-price within small sub-periods. This algorithm can not only detect gradual jumps, but also mathematical jumps, which is the main difference from other algorithms.

The Gumbel test algorithm, proposed by Palmes and Woerner [?], builds idea of extreme value theory. The corresponding test statistic is based that the maximum of increments of log-price converges to Gumbel distribution.

All three algorithms were applied to a dataset of three DAX stock indices, i.e. Adidas, Lufthansa and Deutsche Bank from October 2014 till January 2015. This period covers a financial crisis in Europe, which may result in jumps in process.

The empirical results show that the number of jumps detected by the highs and lows algorithm far exceeds that by the other two algorithms. This phenomenon is caused by the existence of gradual jumps. In comparison with the Gumbel test algorithm, the dynamic algorithm shares a similar performance. Finally, Monte Carlo simulations are proceeded to evaluate the effectiveness of three corresponding algorithms for jump detection.

Keywords: jumps, high frequency data, jump dynamics, intradaily highs and lows, Gumbel distribution, mathematical jumps, gradual jumps

Zusammenfassung

Die Analyse der Preissprünge in einem stetigen Diffusionsprozess unter Verwendung von hochfrequenten Datensätzen auf großes Interesse wegen ihrer wichtigen Rolle in der modernen Finanzmodellierung angezogen. In dieser Arbeit werden drei verschiedene Verfahren dargestellt, um Preissprünge entzudecken. Die drei entsprechende Verfahren sind das dynamische Verfahren, das Hochs und Tiefs Verfahren und das Gumbel-Test Verfahren.

Das dynamische Verfahren wurde von Lee und Mykland [?] entwickelt, der sich auf die Sprung-Dynamik und ihre Verteilung konzentriert. Gleichzeitig können auch die Richtung und die Lage der Sprünge abstimmen. Es wurde empirisch und theoretisch nachgewiesen, dass die Fehlklassifikation der Preissprnge ist vernachlässigbar unter der Verwendung der hochfrequenten Datensätzen.

Das zweite Verfahren, das Hochs und Tiefs Verfahren, basiert auf der entwickelten Theorie von Kloessner [?]. Die Teststatistik von diesem Verfahren wird sich auf die höchsten und tiefsten log-Preise in kleinen Teilperioden aufgebaut. Dieses Verfahren kann nicht nur die graduelle Sprünge, sondern auch die mathematische Sprünge entdecken.

Das Gumbel-Test Verfahren wurde von Palmes and Woerner [?] entwickelt, der basiert auf die Extremwerttheorie. Die Teststatistik dieses Verfahren wird auf die Konvergenz der Maxima von logarithmischen Aktienpreises zu Gumbel Verteilung abgestellt.

Die drei Verfahren wurden auf einen Datensatz des DAX angewandt, welcher den Preisverlauf von drei Aktien: Adidas, Lufthansa und Deutsche Bank im Zeitraum Oktober 2014 bis Januar 2015 abdeckt, und damit die Auswirkungen der Eurokrise in Griechenland verdeutlicht. Es ist sinnvoll vorherzusagen, Preissprünge zu existieren in diesem Zeitraum.

Es ist deutlich von der empirischen Ergebnisse, dass die Anzahl der Sprünge durch das Hochs und Tiefs Verfahren die von den anderen Verfahren überschritt bezüglich der Existenz der graduelle Sprünge. Vergleich mit dem Gumbel Test Verfahren, das dynamische Verfahren führte in ähnlicher Weise aus. Zuletzt wurde die Ergebnisse mittels einer Simulationsstudie verifiziert, um die Wirksamkeit der drei entsprechenden Verfahren zur Entdeckung der Sprünge zu vergleichen.

Schlagwörter: Sprünge, hochfrequente Datensätze, Preisbewegungen, Hochs und Tiefs im Tagesverlauf, Gumbel Verteilung, mathematische Sprünge, graduelle Sprünge

Contents

1 Introduction

In the last two decades, stochastic volatility models have been developed to reveal the principle that the volatility and the correlation of assets changed over time [?]. Continuous stochastic volatility models are widely applied to financial applications, e.g. derivative pricing, bond pricing, and risk management. However, according to enormous empirical findings, it has become increasingly evident that only a limited set of asset returns is suitable to be modeled in a pure diffusion process. Therefore, reasonable specifications of underlying asset price process must be taken into account, such like incorporation of jumps. The idea of incorporation of jumps within a diffusion process was first presented by Merton [?]. The accuracy of empirical results heavily depends on the setting of assumed models, since parametric models were used in Merton's paper. However, due to the limitation of discrete data in continuous stochastic models, the empirical findings show the difficulty of identifying jumps. Due to the benefit of the progress of data storage technologies, the transaction data can be obtained tick-by-tick, which brings the estimation of jumps into a new stage.

Consequently, the detection of jumps in stochastic volatility models has become an important issue in econometrics. For detecting of jumps, the analysis of quadratic variation, which is calculated by summing up all squared discrete returns over very small time periods, is a key subject in this thesis. According to the probability theory, the quadratic variation can be separated into a smooth diffusion component and a jump component [?]. As proposed by Andersen et al. [?], the realized variation including the jump component approximates the quadratic variation. Barndorff-Nielsen and Shephard [?] introduced another measure of price volatility, so-called bipower variation, which is constructed by summing up cross-products of scaled high frequency absolute returns. The bipower variation approximates the diffusion component and has robust performance in the presence of jumps. Barndorff-Nielsen and Shephard [?], [?] compared asymptotic behaviors of the realized variation and the bipower variation for identification of jumps in a fully non-parametric model.

In this work, three different algorithms are under discussion:

- the dynamic algorithm
- the highs and lows algorithm
- the Gumbel test algorithm

The dynamic algorithm was developed by Lee and Mykland [?] and is meant to be applicable to high-frequency observations. Lee and Mykland [?] observed the jumps irregularly occur in the financial markets and believed that it is probably related to the release of news. To obtain a robust test of irregular jump arrivals, it is important to build a dynamic technique to detect jumps, for searching corresponding market information in time. Therefore, with a non-parametric model setting, this algorithm provides a tool for characterizing jump dynamics in each asset price process. Furthermore, the jump size and the sign of jump can be obtained as a by-product, allowing the characterization of jump size distribution and stochastic jump intensities. Moreover, Lee and Mykland [?] mentioned that the misclassification of jumps becomes relative tiny when high-frequency returns are used in the test.

The highs and lows algorithm is based on the work of Kloessner [?], which includes intradaily highs and lows within a subperiod for constructing test statistics. By this algorithm, gradual jumps are distinguished from mathematical jumps. A mathematical jump means a quick reaction to some new market information, while a gradual jump takes several minutes to reach its new equilibrium. According to Zhang et al. [?], Hasbrouck [?], Hansen and Lunde [?], Bandi and Russel [?], the existence of micro-structure noise becomes an obstacle to detect gradual jumps at high sampling frequencies. Therefore, the consideration of this noise in the estimation is necessary. Kloessner [?] empirically and theoretical showed that a sampling frequency high than five minutes can lead to the jump detection test ineffectively. In addition, the positive and the negative jumps can be separately identified by two different specified test statistics with significant results using this algorithm.

The Gumbel test algorithm is based on a recent work of Palmes and Woerner [?], which builds the idea of extreme value theory. The Gumbel test algorithm is different from traditional algorithms that use the difference of the limiting behaviors of realized variation and bipower variation to detect jumps. The test statistic of the Gumbel test algorithm is constructed on the basis that the maximum of price increments asymptotically follows Gumbel distribution. Using this algorithm, it is not only possible to detect jumps, but also to distinguish positive jumps from negative jumps. Since only maximum increments of a Brownian semi-martingale process follows Gumbel distribution, the return process must be set to minus for detecting negative jumps.

An intradaily database on three DAX stock indices of Adidas, Lufthansa and Deutsche Bank from October 2014 till January 2015 is taken for estimating jumps in a return process. This period covers European debt crisis, which can lead to jumps on some specific days. A thorough investigation of the return process is conducted, focusing on the distributional characteristics of both jump intensity, robustness to confidence level and jump size. Finally, Monte Carlo simulations are conducted to compare the effectiveness of three corresponding algorithms for jump detection. Both of spurious detection rate and probability of success in detecting an actual jump are calculated to evaluate the detection accuracy of each algorithm.

This thesis is organized as follows. In Chapter 2, the underling continuous jump diffusion process was presented and three corresponding algorithms, e.g. the dynamic algorithm, the highs and lows algorithm and the Gumbel test algorithm are described in detail. Chapter 3 provides the source of high frequency data and a data cleaning. Chapter 4 presents the empirical results of jump detection using the managed database to the three algorithms. Chapter 5 investigates the performance of each algorithm by a group of simulations. Finally, a conclusion is provided in Chapter 6.

2 Modeling the high frequency data

This work analyzes the performance of three different algorithms on jump detection. The first one is called dynamic algorithm developed by Lee and Mykland [?], whose test statistic is constructed by sequential moving average instantaneous volatility. The second one is referred to highs and lows algorithm proposed by Kloessner [?], which includes new information of the highest and lowest returns in subperiod to detect jumps. The last one is the Gumbel test algorithm, introduced by Palmes and Woerner [?], builds the idea based on the extreme value theory.

It is well known that a stochastic diffusion process is widely used to describe the evolution of asset returns over time [?]. Consequently, a growing interest towards discontinuities in stock price have been seen recently, the so called jumps. As introduced by Merton [?], the arrival of normal information can be modeled as a diffusion process, while the arrival of abnormal information can be modeled as a Poisson process. Although the stock prices can be reflected more accurately modeled by a jump-diffusion process, it leads to incompletion of the financial market in return. The reason is likely that jumps in the stock price cannot be hedged using traded securities. Consider a continuous-time jump-diffusion process for the asset log-price first, and it is conveniently characterized in stochastic differential equation as

$$dp(t) = \mu(t)dt + \sigma(t)dW(t) + J(t)dP(t). \tag{1}$$

$p(t) = \log S(t)$, where $S(t)$ is the spot price of the corresponding asset. The drift $\mu(t)$ denotes a continuous mean process with finite variance and the volatility $\sigma(t)$ denotes a non-negative càdlàg process to allow for occasional jumps. $W(t)$ denotes a standard Wiener process, while a Poisson process $P(t)$ with constant intensity $\lambda \geq 0$ is independent of $W(t)$. In addition, $J(t)$ denotes the jump size, characterized by $J(t) = p(t) - p(t^-)$, with the normal notation $p(t^-) = \lim_{s \uparrow t} p(s)$.

Supposing that there are $m+1$ intradaily observations $S(t)$. As noted previously, $p(t)$ can be computed by the logarithm of the spot price $S(t)$ at discrete times $0 = t_0 < t_1 < \cdots < t_m = T$. For simplicity, it is assumed that a set of $m+1$ intradaily spot prices are available at equally spaced intervals of length $\Delta t = T/m = t_i - t_{i-1}$, $1 \leq i \leq m$. In the non-equidistant case, the simplified assumption can be generalized by setting $\max_i(t_i - t_{i-1}) \to 0$. This assumption is imposed throughout this work.

The *Quadratic Variation* (QV) of a return process is specified as a widely used volatility measure by summing the squared values of the changes of the returns sampled at a sequence of times. QV has many important financial applications, such as Brownian motion and other martingales. QV is well defined for all càdlàg semi-martingales and the formula is obtained by

$$QV = \lim_{m\to\infty} \sum_{i=1}^{m} \{p(t_i) - p(t_{i-1})\}^2, \quad i = 1, 2, \ldots, m. \tag{2}$$

It is straightforward that *Realized Variation* (RV) is a consistent estimator of the corresponding QV, for all semi-martingales [?]. The intradaily return is set as $r_i = p(t_i) - p(t_{i-1}),\ i = 1, 2, \ldots, m$. Then the RV is specified as

$$RV^{(m)} = \sum_{i=1}^{m} \{r_i^{(m)}\}^2, \quad i = 1, 2, \ldots, m. \tag{3}$$

The work of Barndorff-Nielsen and Shephard [?] is focused on the fact that the RV provides arbitrarily good approximations to the QV for a sufficiently large m. The reason to put the number m in the notation of $RV^{(m)}$ is to emphasize the dependence of the estimator on the time grid of intradaily returns.

Recently the availability of high frequency data on financial markets has motivated a huge number of publications devoted to measurement of *Integrated Volatility* (IV) [?]. According to the study of Andersen and Bollerslev [?] and Barndorff-Nielsen and Shephard [?] under the pure diffusion assumption, each r_i conditional on QV is normally distributed with IV. IV contains the information that all comes from the continuous part of a return process. As proposed by Andersen et.al [?], $RV^{(m)}$ converges uniformly in probability to IV in a pure diffusion process, as the sampling frequency of returns approaches infinity. However, when the return process is incorporated with jumps, $RV^{(m)}$ converges to the total price variation QV, rather than IV. As known, QV can be decomposed into IV and jump risk SSJ in the presence of jumps.

$$QV = IV + SSJ, \tag{4}$$

$$IV = \int_{t-1}^{t} \sigma^2(s)ds, \tag{5}$$

$$SSJ = \sum_{0\leq s\leq T} \{J(s)\}^2. \tag{6}$$

where IV contains information about the contribution of the continuous part of the return process to the volatility and SSJ is the sum of the $J(t)$ squared jump sizes observed between 0 and T, representing the discontinuous part of the log-price process.

Followed by Lee and Mykland [?], high spot volatility of returns can be observed sometimes even if there is no jump. At that time, RV can not be used to identify jumps, as its value without a jump may be as high as that with a jump. In order to judge the existence of jumps, instantaneous volatility is used to measure the local variation from the continuous part of the process. A commonly used estimator of instantaneous volatility is the *Realized Bipower Variation* (BPV), which is also suggested and shown as a consistent estimator of IV by Barndorff-Nielsen and Shephard [?]. BPV is defined as

$$BPV^{(m)} = \frac{\pi}{2}\frac{m}{m-1}\sum_{j=2}^{m}|r_j||r_{j-1}|. \tag{7}$$

Different from $RV^{(m)}$, even in the presence of jumps, $BPV^{(m)}$ converges to IV as m goes to infinity.

$$BPV^{(m)} \xrightarrow{p} IV \ as\ m \to \infty, \tag{8}$$

The result implies BPV can be used as a robust technique for the instantaneous volatility estimation even in the presence of jumps.

2.1 The dynamic algorithm

In this section, the dynamic algorithm proposed by Lee and Mykland [?] is in discussion. Compared with the existing nonparametric algorithms for detecting jumps by Barndorff-Nielsen and Shephard [?] (hereafter BNS) and Jiang and Oomen [?] (hereafter JO), the dynamic algorithm was proved of a better effect. The main differences between dynamic algorithm and the other two algorithms (BNS and JO) are described in the following text. The dynamic algorithm measures the performance of jump detection rate by probability of global success in detecting actual jumps and global spurious detection of jumps in one day. However, BNS algorithm takes the difference between QV and BPV to identify the existence of jumps in a time interval, while JO algorithm is based on the similar detection technique of jumps with only one difference. In stead of BPV, JO algorithm uses cumulative delta-hedged gain or loss of a variance swap replicating strategy.

In order to build the jump detection statistic $\mathcal{L}$, the basic intuition should be understood first. This algorithm is constructed based on the fact that the difficult and infrequency Poisson jumps can accurately and dynamically be detected using high frequency data. To proceed by this algorithm, a single time point t_i is taken into consideration and no assumptions about the existence of jumps before or after t_i are made. When the simultaneous tests of more than one hypothesis are involved, a multiple test can be used to solve this problem. In addition, the dynamic algorithm is focused on a dynamic jump detection mechanism in a time interval. Therefore, it is necessary to imagine the stock prices evolving continuously over time with equal time intervals. Supposing a fixed horizon T, $m+1$ observations $0 = t_0 < t_1 < \cdots < t_m$ in $[0, T]$ with distance $\Delta t = T/m$ and one jump occurrence at some time t_i.

As mentioned before, BPV is an excellent consistent estimator of the instantaneous volatility that is included in the denominator of the test statistic. Meanwhile, the test statistic involved with BPV is still robust, even in the presence of jumps. Instead of a fixed window size, the dynamic algorithm for jump detection uses a varying window size K that depends on the sampling frequency of data to construct BPV. Furthermore, the selection of the window size K is important and relates directly to the effectiveness of this algorithm. In each window, $K-1$ observations of stock returns before a testing time t_i are included. If the window size K is too small, the effect of jumps cannot be ignored when estimating an instantaneous volatility and the test statistic is no longer robust in the presence of jumps. Otherwise, if the window size K is too large, the computational burden increases without any marginal benefits. According to the proposal by Lee and Mykland [**?**], the integers between $\sqrt{252 \times \text{nobs}}$ and $252 \times$ nobs are the candidates for K. It is recommended that the smallest integer K within a necessary range is taken as an optimal window size. For example, the window sizes for one-week, one-day, one hour, 30-minute, 15-minute, and 5-minute are 7, 16, 78, 110, 156 and 270, respectively.

Hence, the test statistic $\mathcal{L}(i)$ is determined followed by Lee and Myland [**?**], testing whether there was a jump from t_{i-1} to t_i, and is defined as

$$\mathcal{L}(i) \equiv \frac{r_i}{\widehat{\sigma}(t_i)}, \tag{9}$$

where

$$\widehat{\sigma}(t_i)^2 \equiv \frac{1}{K-2} \sum_{j=i-K+2}^{i-1} |r_j||r_{j-1}|. \tag{10}$$

In order to explain the asymptotic behavior of the proposed statistic, the discussion of the jump detections at the testing time t_i focuses on two conditions, in the absence of jumps and in the presence of jumps. If there is no jumps in the time interval $(t_{i-1}, t_i]$, the asymptotic distribution of the proposed statistic is provided in Theorem 1.

Theorem 1. *Let $\mathcal{L}(i)$ be as definition and $K = \mathcal{O}(\Delta t^{\alpha})$, where $-1 < \alpha < -0.5$. Suppose the stock price follows (1) and all the assumptions are satisfied. Let $\mathcal{A}_n$ be the set o $i \in \{1, 2, \ldots, n\}$. Then as $\Delta t \to 0$,*

$$\sup_{i \in \mathcal{A}_n} |\mathcal{L}(i) - \widehat{\mathcal{L}}(i)| = \mathcal{O}(\Delta t^{\frac{3}{2} - \delta + \alpha - \epsilon}), \tag{11}$$

where δ satisfies $0 < \delta < \frac{3}{2} + \alpha$ and $\widehat{\mathcal{L}(i)} = \frac{U_i}{c}$. Here U_i is a standard normal variable and c is a constant, which equals $\sqrt{2}/\sqrt{\pi} \approx 0.7979$, which implies that $\widehat{\mathcal{L}}(i)$ is a normal distribution.

According to the **Theorem** 1, it is clearly indicated that the corresponding test statistic for jump detection follows approximately the distribution $\widehat{\mathcal{L}}(i)$, which is normally distributed with mean 0 and variance $\frac{1}{c^2}$. Although BPV is replaced with RV to estimate the instantaneous volatility, **Theorem** 1 yields the same result in the absence of jumps. However, the existences of jumps in earlier or later time periods have not been taken into account in this algorithm. For the robustness to the earlier jumps in detecting the current jumps, RV is not suitable in this case.

Supposing that a jump occurs at any time $\tau \in (t_{i-1}, t_i]$. **Theorem** 2 shows a difference performance of the same test statistic, when a jump is involved in the time interval $(t_{i-1}, t_i]$.

Theorem 2. *Let $\mathcal{L}(i)$ be as definition and $K = \mathcal{O}(\Delta t^{\alpha})$, where $-1 < \alpha < -0.5$. Suppose the stock price follows (1) and all the assumptions are satisfied. Suppose there is a jump in time interval $(t_{i-1}, t_i]$. Then,*

$$\mathcal{L} \approx \frac{U_i}{c} + \frac{Y(\tau)}{c\sigma\sqrt{\Delta t}}, \tag{12}$$

where $Y(\tau)$ is actual jump size at actual jump time τ. It is found that the statistic $\mathcal{L}$ becomes very large as the sampling interval $\Delta t \to 0$, which indicates the main difference of the test statistic in the two different conditions.

Compared with the performance of test statistic in **Theorems** 1, the corresponding test statistic presents totally different limiting behaviors in **Theorems** 2, as a jump occurs within the time interval $(t_{i-1}, t_i]$. Overall, if there is no jumps in $(t_{i-1}, t_i]$, the test statistic $\mathcal{L}$ approximately follows a normal distribution. Otherwise, the test statistic $\mathcal{L}$ goes to infinity when $\Delta t \to 0$. Therefore, an appropriate rejection region should be selected for the purpose of jump identification. The asymptotic distribution of the maximums of the test statistics is a nice guider to choose the relevant thresholds for the rejection region. **Lemma** 1 describes the limiting distribution of maximums as

Lemma 1 *Let $\mathcal{L}(i)$ be as definition and $K = \mathcal{O}(\Delta t^{\alpha})$, where $-1 < \alpha < -0.5$. Suppose the stock price follows (1) and all the assumptions are satisfied. Suppose there is no jump in $(t_{i-1}, t_i]$. Then as $\Delta t \to 0$,*

$$\frac{\max_{i \in \mathcal{A}_n} |\mathcal{L}(i)| - C_n}{S_n} \to \xi, \tag{13}$$

where ξ has a cumulative distribution function $P(\xi \leq x) = \exp(-e^{-x})$,

$$C_n = \frac{(2\log n)^{1/2}}{c} - \frac{\log \pi + \log(\log n)}{2c(2\log n)^{1/2}} \quad \text{and} \quad S_n = \frac{1}{c(2\log n)^{1/2}}, \tag{14}$$

where n is the number of observations.

According to **Lemma** 1, the rejection region of the test statistic changes with different significance levels. For instance, when the significance level is set to 1%, the threshold for $\frac{|\mathcal{L}(i)|-C_n}{S_n} = \beta$ follows $P(\xi \leq \beta) = \exp(-e^{-\beta} = 0.99)$. Then β can be calculated as $-\log(\log(0.99)) = 4.6001$. Therefore, if $\frac{|\mathcal{L}(i)|-C_n}{S_n} > 4.6001$, the null hypothesis is rejected, indicating no jumps at the testing time t_i.

In the work of Lee and Mykland [**?**], the probability of misclassification is demonstrated negligible at high sampling frequency. Misclassification can be classified as four types: two types for a single testing time, *failure to detect actual jump* and *spurious detecting of jump* and the two other types of global misclassification over the whole time horizon $[0, T]$, *global failure to detect actual jump* and *global spurious detection of jump*. Lee and Mykland [**?**] have theoretically and empirically showed that both of the conditional and unconditional probabilities of four tpyes of misclassification approach zero.

In summary, for the study of the jump detection, the dynamic algorithm can detect jumps dynamically and determine the exact jump arrival timing. Furthermore, it is reasonably assumed that there is only one jump, when the test statistic is significant. Then, the height of changes in return dominates the jump size and the direction of changes indicates the sign of jump. Thus, the jump size and the sign of jumps can be obtained accordingly at the same time. The process to detect jumps by the *dynamic algorithm* is illustrated in Figure **??**.

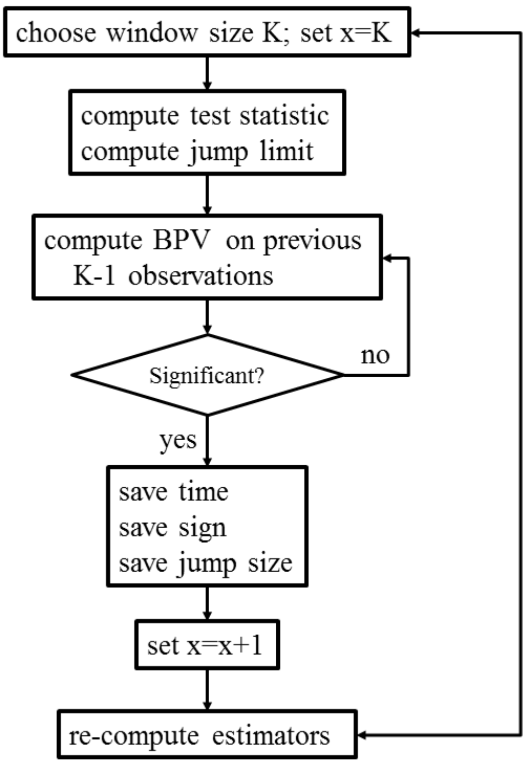


Figure 1: Flowchart of the dynamic algorithm.

2.2 The highs and lows algorithm

In the following, the highs and lows algorithm is introduced, the idea of which first proposed by Andersen [?]. Then, Ane and Metais [?] developed this algorithm slightly. On this basis, Kloessner [?] introduced a new algorithm for jump detection based on sparse sampling in combination with intradaily OHLC data. Kloessner [?] believed that using existing jump detection algorithms are not able to identify gradual jumps, because the sampling distribution with finite sample size is severely affected by high volatility of volatility. A gradual jump that is more realistic in financial markets is distinguished from a mathematical jump. A mathematical jump has fast reactions to the news, which indicates the price moves rapidly from one point to another. But for a gradual jump, the prices are allowed to adjust several minutes to reach its new equilibrium by taking intermediate values. Consequently, both kinds of jumps are important to investigate in the process of jump detections.

According to a statistician's principle of using all available information about the data, some new information is included in this algorithm, such as the highest and lowest log-price during a subinterval $[t_{i-1}, t_i)$. To investigate the influence of micro-structure noise caused by ultra high sampling frequencies, several different data frequencies can be selected for the purpose of comparison.

Followed by Christensen and Podolskij [?] and Kloessner [?], the highest price (highs) and lowest price (lows) are obtained as

$$(p^*)_{i,m} := \sup_{t_{i-1} \leq t \leq t_i} p(t) \quad \text{and} \quad (p_*)_{i,m} := \inf_{t_{i-1} \leq t \leq t_i} p(t). \tag{15}$$

Then, all the open-high-low-close log-prices can be computed in every subinterval $[t_{i-1}, t_i)$, namely the OHLC data.

- opening price $p(t_{i-1})$
- closing price $p(t_i)$
- highest price $(p^*)_{i,m}$
- lowest price $(p_*)_{i,m}$

Kloessner [?] mentioned that the OHLC data can be clearly displayed by the so-called Japaneses candlestick plot, which provides an easy-to-decipher picture of price action. As described in the part (a) of Figure ??, the green rectangular box represents a upward movement, when the close price is greater than the open price. Otherwise, the box shadow is in red color, standing for a downward movement. The upper wick is drawn atop of the rectangle, whose upper coordinate is given by the highest price in the subinterval, while the lower wick is beneath the rectangle with lower coordinate marked as the lowest price. Moreover,

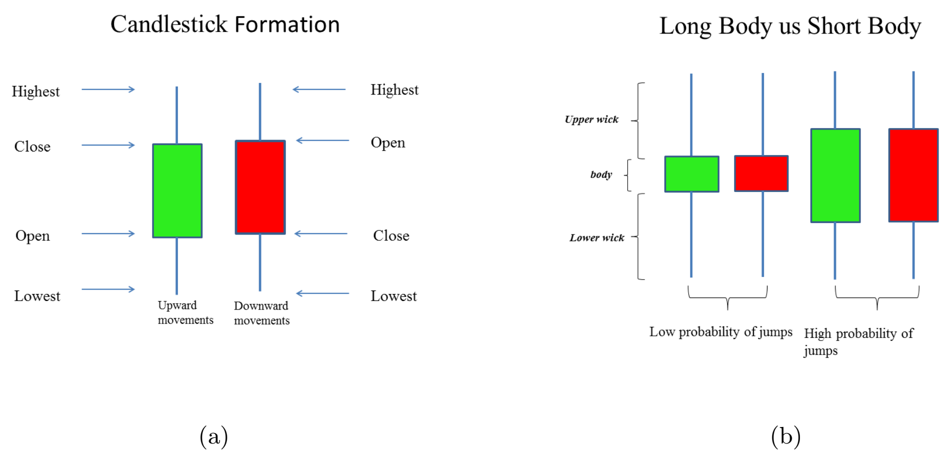


Figure 2: Japaneses candlestick plot. (a): Candlestick formation and (b): Long Body us Short Body

the Japanese candlesticks are particularly suitable to present the diffusive price behavior and jumps. In the part (b) of Figure **??** illustrates the definition of the candlestick's body length $b_{i,m}$, the upper wick's length $uw_{i,m}$ and the lower wick's length $lw_{i,m}$ as

$$b_{i,m} = |p(t_i) - p(t_{i-1})|, \tag{16}$$

$$uw_{i,m} = (p^*)_{i,m} - \max\{p(t_{i-1}), p(t_i)\}, \tag{17}$$

$$lw_{i,m} = \min\{p(t_{i-1}), p(t_i)\} - (p_*)_{i,m}. \tag{18}$$

In the work of Kloessner [**?**] has revealed that the length of a upper wick and a lower wick can be respectively interpreted as the upward diffusive volatility and downward diffusive volatility. Meanwhile, the jump size is measured by the height of a rectangular box. When the upper wick is very large, it means that the highest price within subinterval significantly exceeds the open and close price, i,e the price will drop prior to the end point of subinterval. Compared to a tiny wick, candlesticks with a huge body can be explained by a high probability of jumps. On the other side, a combination of a large wick and a tiny body indicates that it is probable to exist jumps. Based on the above theoretical reasons and empirical results, Kloessner [**?**] used OHLC data to construct a test statistic to identify jumps. The details of analysis are in the following.

According to Equation ??, QV can be decomposed into IV and SSJ. SSJ stands for the discontinuous part of a stochastic jump-diffusion process and can be computed by summing the squared jump sizes $J(t)$ observed between 0 and T. When the $J(s)$ is larger than 0, it is called a positive jump, otherwise, a negative jump. The definition of positive discontinuous part SS_pJ and negative discontinuous part SS_nJ are as follows:

$$SS_pJ = \sum_{J(s)>0} \{J(s)\}^2, \tag{19}$$

$$SS_nJ = \sum_{J(s)<0} \{J(s)\}^2. \tag{20}$$

where $J(s)$ denotes the observed jump size between 0 and T.

As mentioned before, BPV is a consistent estimator of IV, even in the presence of jumps. However, the power of BPV is very poor to detect gradual jumps at a sampling frequency of every one minute or higher, followed by Kloessner [?]. Therefore, new estimators of IV should be considered to construct an effective test statistic for jump detection, with involving high frequency data. Following Barndorff-Nielsen et.al [?], a indicator function $1_{r_{i,m}>0}$ (or $1_{r_{i,m}<0}$) replacing $J(s) > 0$ (or $J(s) < 0$) is used to distinguish the estimators for SS_pJ from SS_nJ. This setting is suitable both for a gradual jump and a mathematical jump. In addition, the upper wick $uw_{i,m}$, the lower wick $lw_{i,m}$ and the product term $uw_{i,m} \cdot lw_{i,m}$ are all taken into consideration to yield a consistent estimator of IV. As a result, the following unbiased estimators are used as an approximation of IV regardless of a jump,

$$4uw_{i,m}^2, \quad 4lw_{i,m}^2, \quad \frac{4}{8\log(2)-5} lw_{i,m} \cdot uw_{i,m}. \tag{21}$$

then, the optimal linear combination of the estimators is

$$\widehat{IV}_{i,m}^{(l)} = 1.3277uw_{i,m}^2 + 1.3227lw_{i,m}^2 + 2.4847uw_{i,m} \cdot lw_{i,m}. \tag{22}$$

This optimal estimator is the one has the smallest variance with $0.7244 \cdot \frac{\sigma^4_{\frac{i-1}{m}}}{m^2}$ among all consistent estimators for IV. Moreover, the squared body length $b_{i,m}^2$ is a robust estimator for SSJ, when a large jump exists in the process. However, if there is no jumps, the $b_{i,m}^2$ is not appropriate to estimate SSJ, since only one estimator b^2 approximately estimate IV. With summing all corresponding estimators, such as $uw_{i,m}^2$ and $lw_{i,m}^2$, then the value is significantly larger than IV. To solve this problem, the resulting estimator of SSJ is defined as

$$\widehat{SSJ}_{i,m}^{(l)} = b_{i,m}^2 - 1.4383uw_{i,m}^2 - 1.4383lw_{i,m}^2 - 2.0605uw_{i,m} \cdot lw_{i,m}, \tag{23}$$

where the mean and the variance of the estimator are essentially 0 and $3.7474 \cdot \frac{\sigma^4_{\frac{i-1}{m}}}{m^2}$.

Naturally, it is also important to distinguish positive jumps from negative ones in the detection process. Similarly, the estimators for the sum of squared positive and negative jumps are defined as,

$$\widehat{SSpJ}^{(l)}_{i,m} = \begin{cases} b^2_{i,m} - 3.2047uw^2_{i,m} - 3.2047lw^2_{i,m} - 3.0301uw_{i,m} \cdot lw_{i,m}, & r_{i,m} > 0, \\ 1.7633uw^2_{i,m} + 1.7633lw^2_{i,m} + 0.9697uw_{i,m} \cdot lw_{i,m}, & r_{i,m} < 0, \end{cases} \quad (24)$$

$$\widehat{SSnJ}^{(l)}_{i,m} = \begin{cases} 1.7633uw^2_{i,m} + 1.7633lw^2_{i,m} + 0.9697uw_{i,m} \cdot lw_{i,m}, & r_{i,m} > 0, \\ b^2_{i,m} - 3.2047uw^2_{i,m} - 3.2047lw^2_{i,m} - 3.0301uw_{i,m} \cdot lw_{i,m}, & r_{i,m} < 0, \end{cases} \quad (25)$$

where the mean and variance of the corresponding estimator is respectively equal to 0 and $4.9360 \cdot \frac{\sigma^4_{\frac{i-1}{m}}}{m^2}$ in both cases, in the absence of jumps.

According to Equations **??**, **??**, **??** and **??**, these estimators for IV and SSJ include only the wicks' lengths for the purpose of robustness of estimation. Particularly, in order to reduce the variance of the estimator significantly in the absence of jumps, the products of body and wicks' lengths are considered to construct new consistent estimators for IV and SSJ. $\widehat{IV}^{(p)}_{i,m}$ is defined as follows:

$$\widehat{IV}^{(p)}_{i,m} = 0.4416(uw^2_{i,m} + lw^2_{i,m}) + 1.3851uw_{i,m} \cdot lw_{i,m} + 1.1809(uw_{i,m} \cdot b_{i,m} + lw_{i,m} \cdot b_{i,m}). \quad (26)$$

Comparing the variance of $\widehat{IV}^{(l)}_{i,m}$ with $0.7244 \cdot \frac{\sigma^4_{\frac{i-1}{m}}}{m^2}$, the new estimator $\widehat{IV}^{(p)}_{i,m}$ has a smaller variance with $0.2921 \cdot \frac{\sigma^4_{\frac{i-1}{m}}}{m^2}$.

In an analogous way, the estimators for SSJ, $SSpJ$ and $SSnJ$ are formulated, respectively, as

$$\widehat{SSJ}^{(p)}_{i,m} = 0.6576(uw^2_{i,m} + lw^2_{i,m}) + 0.5552uw_{i,m} \cdot lw_{i,m} - 2.8089(uw_{i,m} + lw_{i,m}) + b^2_{i,m} \quad (27)$$

with the mean and variance given by 0 and $1.3014 \cdot \frac{\sigma^4_{\frac{i-1}{m}}}{m^2}$ in the absence of jumps, and

$$\widehat{SSpJ}^{(p)}_{i,m} = \begin{cases} b^2_{i,m} + 0.7706(uw^2_{i,m} + lw^2_{i,m}) + 0.7349uw_{i,m} \cdot lw_{i,m} & \\ \qquad -3.1847(uw_{i,m} + lw_{i,m}), & r_{i,m} > 0, \\ -0.1130(uw^2_{i,m} + lw^2_{i,m}) - 0.1842uw_{i,m} \cdot lw_{i,m} & \\ \qquad +0.3758(uw_{i,m} \cdot b_{i,m} + lw_{i,m} \cdot b_{i,m}), & r_{i,m} < 0, \end{cases} \quad (28)$$

$$\widehat{SSnJ}^{(p)}_{i,m} = \begin{cases} -0.1130(uw^2_{i,m} + lw^2_{i,m}) - 0.1842uw_{i,m} \cdot lw_{i,m} & \\ \qquad +0.3758(uw_{i,m} \cdot b_{i,m} + lw_{i,m} \cdot b_{i,m}), & r_{i,m} > 0, \\ b^2_{i,m} + 0.7706(uw^2_{i,m} + lw^2_{i,m}) + 0.7349uw_{i,m} \cdot lw_{i,m} & \\ \qquad -3.1847(uw_{i,m} + lw_{i,m}), & r_{i,m} < 0, \end{cases} \quad (29)$$

where both the approximate mean and variance are equal to 0 and $0.7071 \cdot \frac{\sigma^4_{\frac{i-1}{m}}}{m^2}$,respectively, in the absence of jumps.

According to all the subperiodic estimators, the daily quantities of IV, SSJ, $SSpJ$ and $SSnJ$ estimators can be then constructed as follows,

$$\widehat{IV}^{(l)} = \sum_{i=1}^{m} \widehat{IV}^{(l)}_{i,m}, \qquad (30a) \qquad \widehat{IV}^{(p)} = \sum_{i=1}^{m} \widehat{IV}^{(p)}_{i,m}, \qquad (30b)$$

$$\widehat{SSJ}^{(l)} = \sum_{i=1}^{m} \widehat{SSJ}^{(l)}_{i,m}, \qquad (31a) \qquad \widehat{SSJ}^{(p)} = \sum_{i=1}^{m} \widehat{SSJ}^{(p)}_{i,m}, \qquad (31b)$$

$$\widehat{SSpJ}^{(l)} = \sum_{i=1}^{m} \widehat{SSpJ}^{(l)}_{i,m}, \qquad (32a) \qquad \widehat{SSpJ}^{(p)} = \sum_{i=1}^{m} \widehat{SSpJ}^{(p)}_{i,m}, \qquad (32b)$$

$$\widehat{SSnJ}^{(l)} = \sum_{i=1}^{m} \widehat{SSnJ}^{(l)}_{i,m}, \qquad (33a) \qquad \widehat{SSnJ}^{(p)} = \sum_{i=1}^{m} \widehat{SSnJ}^{(p)}_{i,m}. \qquad (33b)$$

Compared to the standard estimator, Kloessner [?] has proved that $\widehat{IV}^{(l)}$ and $\widehat{IV}^{(p)}$ both performed well at a high or moderate frequency in the Monte Carlo simulation. $\widehat{IV}^{(l)}$ is unbiased for testing all frequency data, while $\widehat{IV}^{(p)}$ suffers a small positive bias at a relative low frequency. On the other side, $\widehat{SSJ}^{(p)}$ outperforms $\widehat{SSJ}^{(l)}$ even at a moderate frequency of 10 or 15 minutes, since $\widehat{SSJ}^{(p)}$ has a considerably large variance for estimating in the absence of jumps. However, $\widehat{SSJ}^{(p)}$ comes under a higher bias than $\widehat{SSJ}^{(l)}$.

$\widehat{IQ}$ is a OHLC-based estimator of $IQ = \int_0^1 \sigma_t^4 dt$, which is defined as follows,

$$\widehat{IQ} = \frac{1}{2}\widehat{IQ}_p + \frac{1}{2}\widehat{IQ}_n + \widehat{IQ}_z, \tag{34}$$

with

$$\widehat{IQ}_p = \sum_{r_{i,m}>0} \frac{16}{3}[\{(p^*)_{i,m} - p(t_i)\}^4 + \{p(t_{i-1}) - (p_*)_{i,m}\}^4], \tag{35}$$

$$\widehat{IQ}_n = \sum_{r_{i,m}<0} \frac{16}{3}[\{(p^*)_{i,m} - p(t_{i-1})\}^4 + \{p(t_i) - (p_*)_{i,m}\}^4], \tag{36}$$

$$\widehat{IQ}_z = \sum_{r_{i,m}=0} \frac{16}{3}[\{(p^*)_{i,m} - p(t_{i-1})\}^4 + \{p(t_i) - (p_*)_{i,m}\}^4], \tag{37}$$

Following the idea of Kloessner [?], the capable of capturing gradual jumps and robust to heavy fluctuations in volatility should be balanced by the highs and lows algorithm. Thus three different test statistics of the highs and lows algorithm are constructed as

$$\widehat{TJ} = \sqrt{m}\frac{\sum_{i=1}^{m} \widehat{SSJ}_{i,m}^{(p)}}{\sqrt{1.3014\widehat{IQ}}}, \tag{38}$$

$$\widehat{TJ}_p = \sqrt{m}\frac{\sum_{i=1}^{m} \widehat{SSpJ}_{i,m}^{(t)}}{\sqrt{0.8602\widehat{IQ}}}, \tag{39}$$

$$\widehat{TJ}_n = \sqrt{m}\frac{\sum_{i=1}^{m} \widehat{SSnJ}_{i,m}^{(t)}}{\sqrt{0.8602\widehat{IQ}}}, \tag{40}$$

with

$$\widehat{SSpJ}_{i,m}^{(t)} = \begin{cases} b_{i,m}^2 + 1.3982(uw_{i,m}^2 + lw_{i,m}^2) + 2.0902uw_{i,m} \cdot lw_{i,m} & \\ \quad -3.968b_{i,m}(uw_{i,m} + lw_{i,m}), & r_{i,m} > 0, \\ 0, & r_{i,m} < 0, \end{cases} \tag{41}$$

$$\widehat{SSnJ}_{i,m}^{(t)} = \begin{cases} 0, & r_{i,m} > 0, \\ b_{i,m}^2 + 1.3982(uw_{i,m}^2 + lw_{i,m}^2) + 2.0902uw_{i,m} \cdot lw_{i,m} & \\ \quad -3.968b_{i,m}(uw_{i,m} + lw_{i,m}), & r_{i,m} < 0, \end{cases} \tag{42}$$

In addition, the discussion of jump arrival time and jump size is also worth conducting. When the test statistic is significant for identifying positive or negative jumps, at least one jump on that day is to be considered. Otherwise, the number of jumps is set to zero. Firstly, the highest value of $\widehat{SSpJ}$ ($\widehat{SSnJ}$) is determined, regarding as the largest jump on that day, where

$$\widehat{SSpJ}_{j,m} = \max_{i=1,2,\ldots,m}(\widehat{SSpJ}_{i,m}), \qquad \widehat{SSnJ}_{j,m} = \max_{i=1,2,\ldots,m}(\widehat{SSnJ}_{i,m}). \tag{43}$$

Secondly, the exact jump arrival timing when the $\widehat{SSpJ}_{j,m}$ ($\widehat{SSnJ}_{j,m}$) occurs is recorded as jump arrival time on that day. For the jump size, it is certainly dominated by

$$J(t_j) = \widehat{SSpJ}_{j,m}, \qquad J(t_j) = -\widehat{SSnJ}_{j,m}, \tag{44}$$

respectively. Thirdly, the $\widehat{SSpJ}_j$ ($\widehat{SSnJ}_j$) is replaced with average value of all other $\widehat{SSpJ}_i$ ($\widehat{SSnJ}_i$), $i \neq j$, on that day. Finally, a new round of calculating the corresponding test statistic continues. Figure **??** illustrates the procedure of jump detection algorithm using the highs and lows of returns, where the second algorithm is call *highs and lows algorithm*.

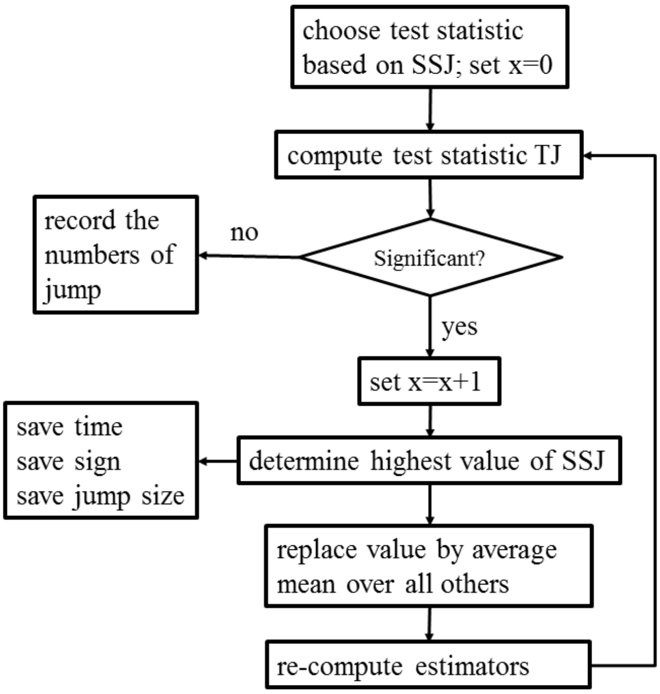


Figure 3: Flowchart of the highs and lows algorithm.

2.3 The Gumbel test algorithm

This section describes the Gumbel test algorithm for jump detection, which is based on extreme value theory. The idea behind is that the maximum of increments of a Brownian semi-martingale process follows Gumbel distribution in the absence of jumps. Using this algorithm not only jumps can be detected, but also the direction of jumps can be determined. In order to detect negatives jumps, the returns must be set to minus before applied to the Gumbel test algorithm, since this algorithm is only valid for maximums, rather than minimums. The main idea using extreme value theory for jump detection was first proposed by Lee and Mykland [?], but it was more restrictive and did not distinguish between positive jumps and negative jumps. Then Palmes and Woerner [?] improved the algorithm on the basis of Lee and Mykland [?], by setting a general drift and jump component, an unrestrained structure correlation between continuous part and jump component within the process and a general path-wise dependent volatility process.

Palmes and Woerner [?] considered a stochastic volatility model for return process without jump components. It is defined in *Itô* semi-martingales of the form,

$$\tilde{p}(t) = \int_0^t \sigma(s)dW(s) + \int_0^t \mu(s)ds = \int_0^t \sigma(s)dW(s) + D(t), \quad 0 \leq t \leq 1, \tag{45}$$

where $\tilde{p}(t)$ denotes the stock return without jump occurrence. $D(t) = \int\limits_0^t \mu(s)ds$ denotes the mean value. $W(t)$, $\sigma(s)$ and $\mu(s)$ are the same with the definition in the beginning of this section. Without loss of generality, a unit time interval $[0,1]$ can take place of an interval $[0,T]$ for some $T > 0$ for the convenience of analysis. Palmes and Woerner [?] also impose some assumptions on the drift and volatility process,

- there are three global constants $0 < V \leq K < \infty$ and $0 < \alpha \leq 1$, and two functions $\alpha : \Omega \to (0,1]$ and $K : \Omega \to (0,\infty]$,
- for each $\omega \in \Omega$, $V \leq \sigma_t(\omega) \leq K$, $0 \leq t \leq 1$,
- for each $\omega \in \Omega$, $|\sigma_t(\omega) - \sigma_s(\omega)| \leq K(\omega)|t-s|^{\alpha(\omega)}$, $0 \leq s,t \leq 1$,
- for each $\omega \in \Omega$, $|\sigma_t(\omega)| \vee |d_t(\omega)| \leq K(\omega)$, $0 \leq s,t \leq 1$, and $t \mapsto d_t(\omega)$ is Lebesgue measurable.

In the following, Palmes and Woerner [?] came up with an idea of two scale time grids, which are different from the other two algorithms. It can be observed $m+1$ numbers of time points at $0, \frac{1}{m}, \ldots, 1$, where setting $m = n^2$ for $n \in \mathcal{N}$. And the sampling times are set to $\frac{l}{n^2}$, $l = 0, 1, \ldots, n^2-1, n^2$ with

$$\frac{l}{n^2} = \frac{kn+j}{n^2} = t_{k,j}, \quad 0 \leq k,j < n. \tag{46}$$

where k and j indicate the coarse and finer parts of the grid on the unit interval respectively. With the finer scale, the increment of return process can be defined as

$$\Delta\tilde{p}(t_{k,j}) = \tilde{p}(t_{k,j} + \frac{1}{n^2}) - \tilde{p}(t_{k,j}) \tag{47}$$

In addition, the volatility notion is also taken some abbreviations,

$$\sigma_{k,j} = \sigma(t_{k,j}), \quad \sigma_k = \sigma(t_{k,0}). \tag{48}$$

Analogous to the dynamic algorithm, Palmes and Woerner [?] used BPV to estimate the spot volatility, resulting in a robust test to detect jumps. The corresponding estimator of σ_k is defined as,

$$\widehat{\sigma}_k^2 = \frac{\pi n^2}{2(n-1)} \sum_{j=0}^{n-2} |\Delta\tilde{p}(t_{k,j})||\Delta\tilde{p}(t_{k,j+1})| \tag{49}$$

Then the test statistic follows straightforward.

Theorem 3. *Set a_m and b_m as follows,*

$$a_m = \sqrt{2\log m}, \quad b_m = a_m - \frac{\log(\log m) + \log(4\pi)}{2\sqrt{2\log m}}, m \in \mathcal{N} \tag{50}$$

and the statistic T_n *is given by*

$$T_n = n \max_{k,j}\{\frac{\Delta\tilde{p}(t_{k,j})}{\widehat{\sigma}_k}\}, \quad n \in \mathcal{N}. \tag{51}$$

Then we obtain the following approximation

$$a_{n^2}(T_n - b_{n^2}) \xrightarrow{d} \mathcal{G}, \quad n \to \infty, \tag{52}$$

where $\mathcal{G}$ *is Gumbel distribution with the cumulative distribution function* $x \mapsto e^{-e^{-x}}$.

It is obvious that the corresponding test statistic approaches standard Gumbel distribution in the absence of jumps, when the sample size is large enough. Moreover, Palmes and Woerner [?] also proved that the normalized test statistic converges to infinity at a certain rate $\sqrt{n}$, when the positive jumps are present. By the Gumbel test algorithm, the exact jump arrival time can not be determined straightforward. However, it is reasonable to assume that the jump occurs when the largest increment appears within the process during a given period and the sample size is dominated by the corresponding increment. After detecting the first jump, the second jump can be similarly detected based on the new sequence only excluding the highest increment on that day. The flowchart of *Gumbel test algorithm* is shown in Figure **??**.

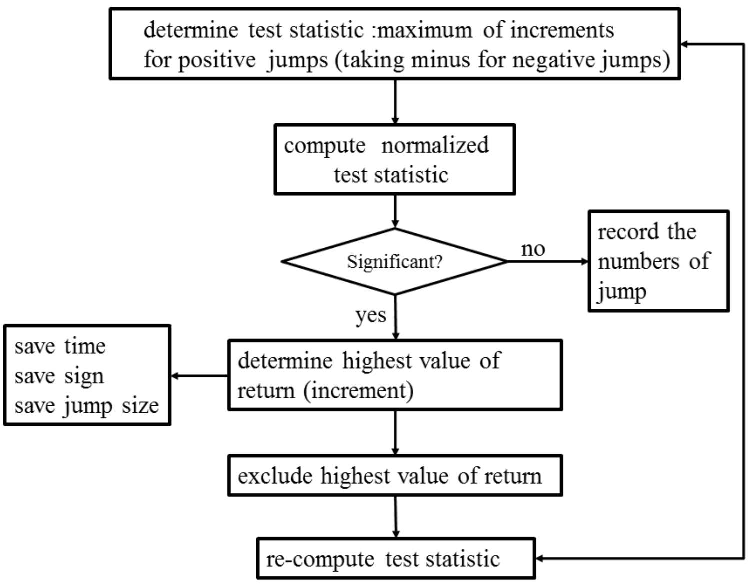


Figure 4: Flowchart of the Gumbel test algorithm.

3 Data and descriptive statistics

From this section the empirical analysis is introduced to study the jump detection problem. All spot prices are downloaded from Bloomberg which provides realtime and historical financial market data and economic data worldwide. In this work, three stocks i.e. *Deutsche Bank*, *Adidas* and *Lufthansa*, from Frankfurt Stock Exchange are analyzed. At the same time, three different frequencies, i.e. one-minute, 5-minute and 10-minute are selected in the empirical analysis from October 2014 till January 2015. In these four months, European debt crisis was ongoing, which may cause jumps in the stock price process. Considering the argument above, the stock return is calculated as $r_t = (\log S_t - \log S_{t-1}) \times 10000$, where the r_t and S_t denote the stock return and the closing spot price in one day, respectively. Stock return is multiplied 10000 in order to state the value bigger and visualize the amount clearly.

In order to promote the application of corresponding data, the data is required to clean properly. The electronic trading platform Xetra trading takes place from $9:00$ until $17:30$ every day except for some special cases, such as the Xetra closed at $14:00$ on December 30^{th} 2014. Then some entries with a time stamp outside $9:00$ and $17:30$ must be deleted for this purpose. Moreover, the missing values should be filled by the last observations for the purpose of time continuity. Hense, a homogeneous time series is created for the following analysis.

Table **??** summaries some basic statistics of the intradaily data at the frequency of one-minute, 5-minute and 10-minute for each stock. The sample sizes at each frequency for estimation are 44968, 9064 and 4576, respectively. It can be observed obviously from the table that the average returns of the three stocks are all around zero. With the highest sampling frequency of 1-minute, the average returns of Adidas and Lufthansa are both positive. However, as the sampling frequency decreases, the average return of Adidas becomes smaller and turns negative, while the value of Lufthansa maintains positive and is larger. For the stock of Deutsche Bank, the average returns are always negative at each frequency. On the other side, the standard deviation of returns measures the volatility of the data. Although the returns are expected to be more volatile at a higher frequency due to the existence of market micro-structure noise in data, the results of the selected three DAX stock returns are presented in opposite. The results from the table show that the standard deviation is very large at the lowest frequency and almost three times of that at the highest frequency. This phenomenon may be caused by the different sample size, since a large sample more closely approximates the population with smaller the standard deviation. Comparing three

Table 1: Descriptive statistics of intradaily returns based on different data frequency.

	Mean	Std.dev	Skewness	Excess kurtosis	J-B	$Q^2(20)$
Panel A. Estimation for 1-minute frequency intradaily data (42413 observstions)						
Adidas	0.001	7.901	8.308	477.229	480610545^c	112.376^c
Lufthansa	0.044	10.105	−0.995	111.577	22009223^c	149.763^c
Deutsche Bank	−0.017	8.967	2.135	94.510	15818301^c	54.839^c
Panel B. Estimation for 5-minute frequency intradaily data (9064 observstions)						
Adidas	−0.002	17.676	5.303	149.222	7974651^c	24.937^c
Lufthansa	0.222	21.559	1.184	31.490	355370^c	49.490^c
Deutsche Bank	−0.084	19.417	0.680	23.137	191416^c	29.967^a
Panel C. Estimation for 10-minute frequency intradaily data (4576 observstions)						
Adidas	−0.018	24.991	3.908	82.722	1242474^c	21.373
Lufthansa	0.433	29.795	1.206	22.593	92916^c	44.946^c
Deutsche Bank	−0.176	26.502	0.759	8.862	14551^c	32.149^b

Notes: 1.*a, b and c denote significance at the* 10%*,* 5% *and* 1% *levels, respectively.* 2.*JB statistics are based on Jarque and Bera (*1987*) and are asymptotically chi-squared distributed with* 2 *degrees of freedom.* 3. $Q^2(20)$ *denotes the LjungBox Q test for* 20*th order serial correlation of the squared returns.*

stock returns at the same frequency, Lufthansa has the largest standard deviation, followed by Adidas and Deutsche Bank. Thus, it is reasonable to expect more jumps detected for Lufthansa than those for Adidas and Deutsche Bank.

In accordance with the coefficient of skewness, almost all returns present the right-skewed features at each frequency, except for Lufthansa at the frequency of every one minute. The negative skewness of Lufthansa means that more negative returns are observed in the process. In regard to the coefficient of excess kurtosis, all values are significantly different from zero, hereby implying that the distribution of returns has larger, thicker tails than that of a normal distribution. The Jarque-Bera normality tests significantly reject the hypothesis of normality for all three stocks regardless of the selected sampling frequency. To test whether any of a group of auto-correlations of the return process are different from zero, the LjungBox $Q^2(20)$ statistics for the three stocks are proceeded. The results show that the test statistics are significantly rejected, which indicates the returns are random and independent.

Figure **??** illustrates the descriptive density graphs for stock of Adidas, Lufthansa and Deutsche Bank at each frequency. In addition, the density of a normal distribution is complemented to compare with the density of the underlying return process. Followed by Xiong and Idzorek [**?**], almost all empirical results present that most asset returns are not normally distributed and their distributions are skewed to the left (or occasionally the right) of the mean value. In addition, most asset return distributions are more leptokurtic, or fatter tailed, than normal distributions. As expected, the density of the underlying return process for three stocks are shown from the figures totally different from that of normal distribution. Compared with normal distribution, the density function of the underlying return process are flatter and have heavier tails.

In Figure **??**, an overview of spot price and return process during the period of October 2014 to January 2015 is presented for each stock based on 1 minute sampling frequency. At the beginning of October 2014, the spot prices of the three stocks dramatically moved downward due to the impact of the European debt crisis. Then, the spot price of Adidas began to increase until reaching its peak at the end of November 2014, and then decreased in December 2014. Deutsche Bank showed a similar behavior in that period. Furthermore, rising trends of these two stocks can be predicted after January 2015. For the stock of Lufthansa, the spot price continued rising after its first small falling. Moreover, some positive and negative peaks can be observed from the pattern of return process for each stock, which indicates the rapid price movements and existence of jumps. Generally, Adidas and Deutsche Bank gathered more positive peaks than negative ones in the return process. In contrast, Lufthansa had more large negative peaks during that period.

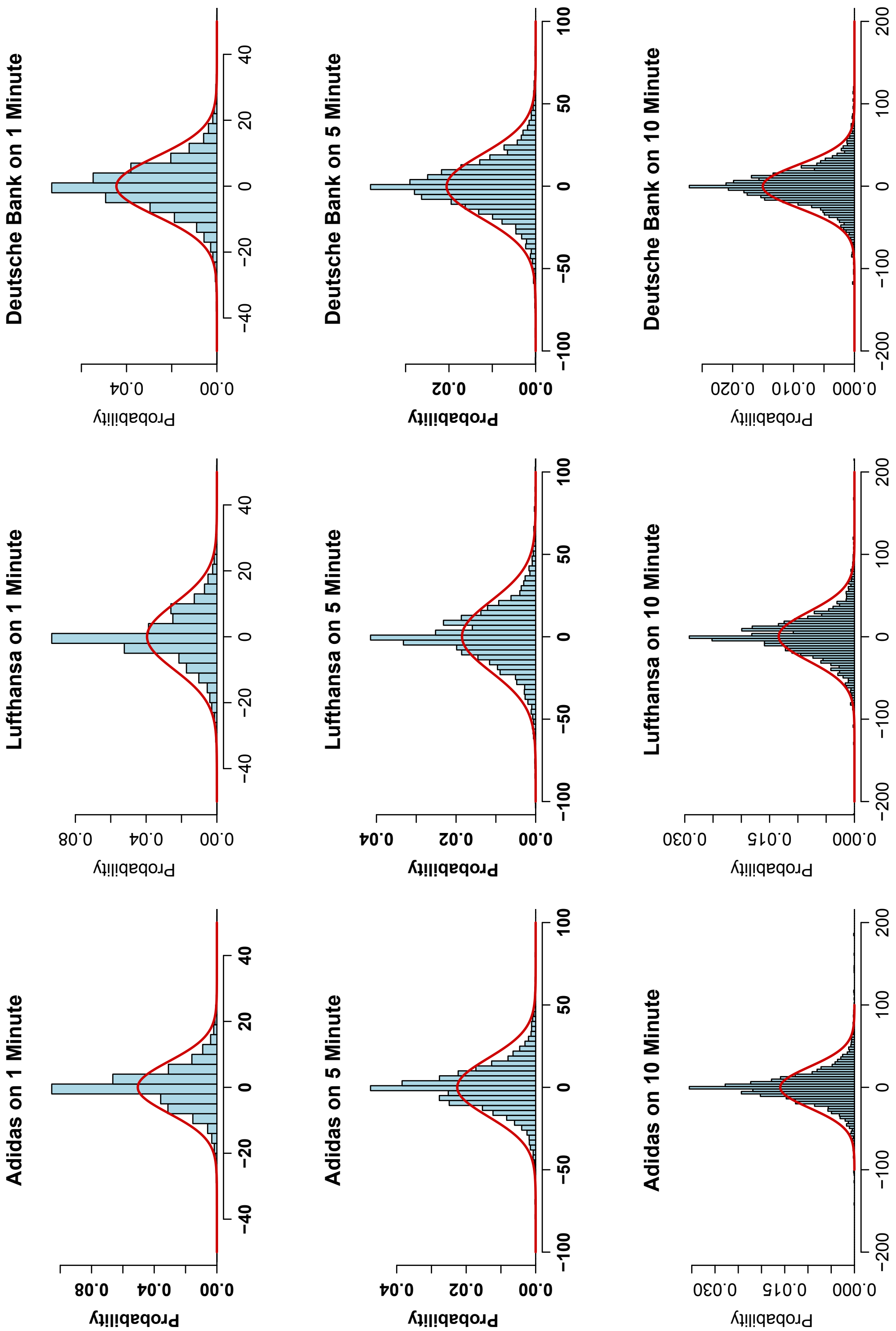


Figure 5: Density of intradaily returns against normal distribution at the frequency of every 1-minute, 5-minute and 10-minute for stocks of Adidas, Lufthansa and Deutsche Bank.

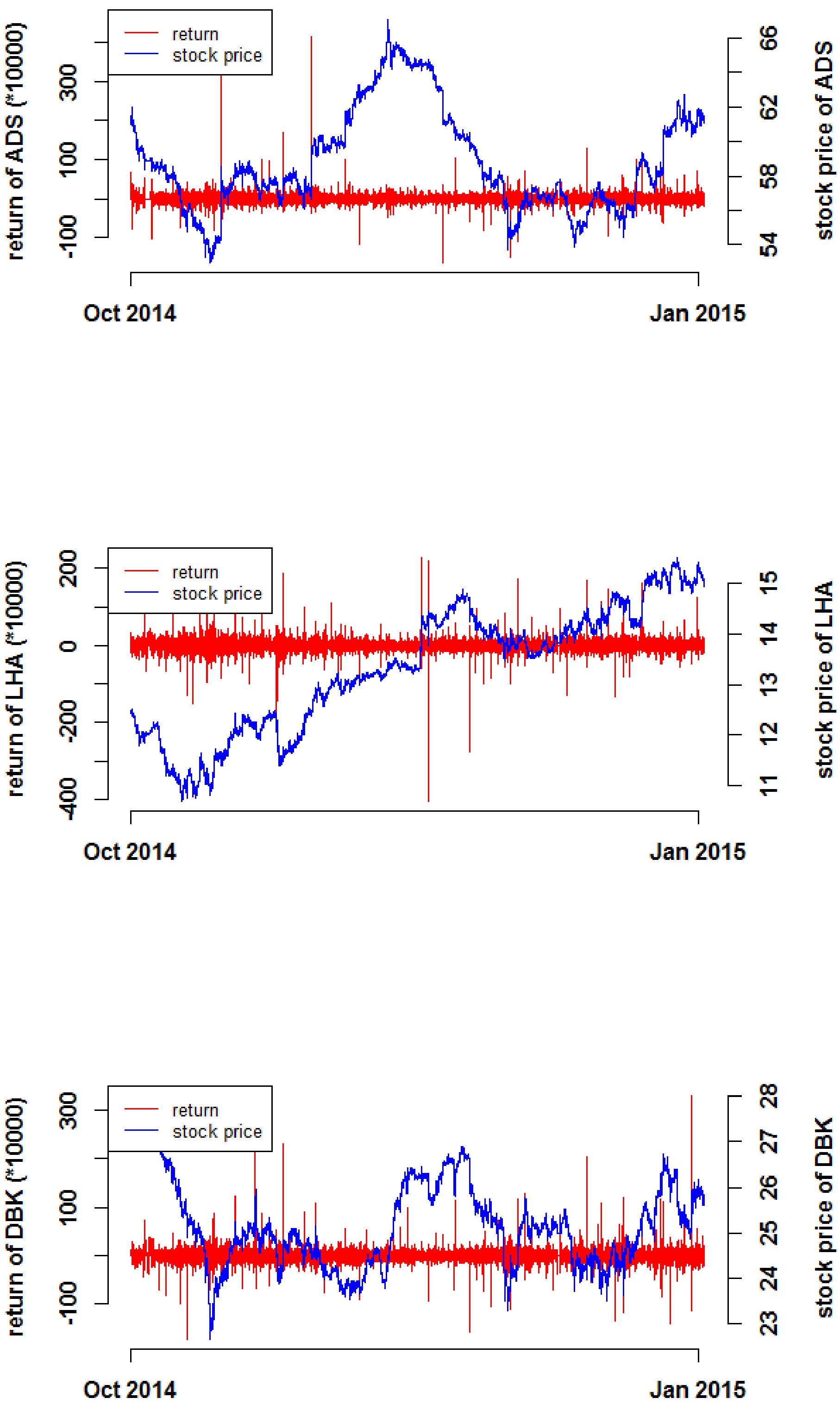


Figure 6: Stock price (blue) and stock returns (red) of Adidas, Lufthansa and Deutsche Bank from October 2014-January 2015 at the sampling frequency of one minute.

4 Empirical Analysis

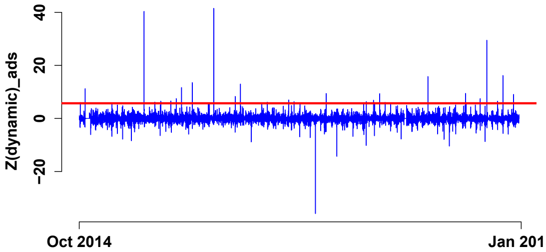
This section discusses the empirical result using the three corresponding algorithms. With heavily relying on the sample size and the data frequency, it is better to compare the empirical results of test statistics under three cases with different data frequencies. In other literature, 5-minute returns are most widely used as the best sampling frequency for analysis. However, the test statistics in this work are based on asymptotic results and sensitive to the frequency. Therefore, three different sampling frequencies of every 1, 5 and 10 minutes are taken into account in the following empirical analysis.

4.1 Comparison of test statistics

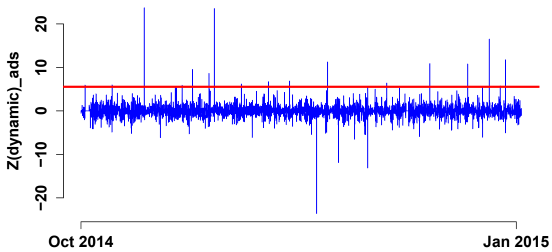
In order to evaluate the effectiveness of the three algorithms, it is natural to start with calculation of the test statistics at different data frequencies and then compare their values with critical value corresponding to a given significance level. Figures **??**, **??** and **??** illustrate the pattern of test statistics of the three algorithms according to different sampling frequencies at 5% significance level. It can be seen that the performance of the three algorithms for each stock is comparable and similar. In addition, the pattern of the value of test statistic matches that of the return process for each stock that has been presented in the last section. In comparison, the values of test statistic using 10-minute returns are smaller than those at 5-minute sampling frequency for all three stocks. This result indicates that less jumps are detected with lower data frequency. On the other side, the test statistic using highs and lows algorithm is much higher than that using the other two algorithms, which is probably caused by gradual jumps. As noted in the introduction of this algorithm, highs and lows algorithm is the only one that is able to detect to gradual jumps. Considering the ultra high values of test statistic of highs and lows algorithm, it is likely to exist the overestimation of jumps, which would be discussed further in next section.

4.2 Number of jumps and jump intensity

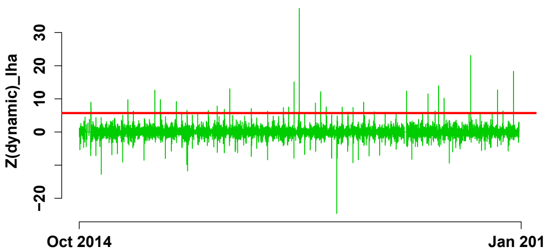
Naturally, the previously discussed algorithms are applied to the three stock indices to calculate the number of daily jumps. As mentioned before, the positive and the negative jumps should be treated separately, since they have different impacts on equity and derivative pricing analysis. In Table **??**, the exact number of jumps between October 2014 and January 2015 is summarized. With this exact jump count on hand, the jump intensity for each stock is refined in the same table by dividing the number of detected jumps by the sample size. It



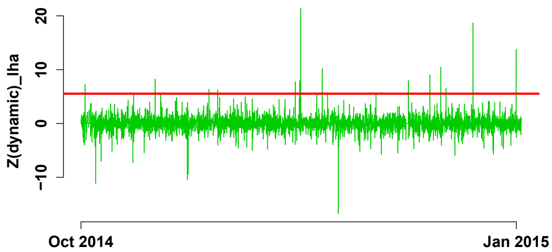
(a) Dyna. Algo on 5 minute



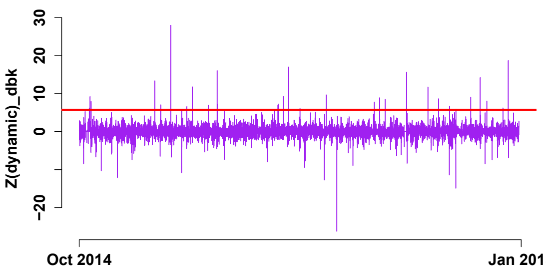
(b) Dyna. Algo on 10 minute



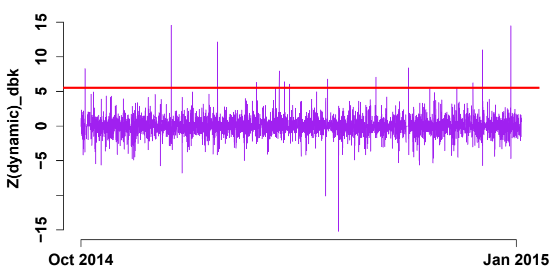
(c) Dyna. Algo on 5 minute



(d) Dyna. Algo on 10 minute

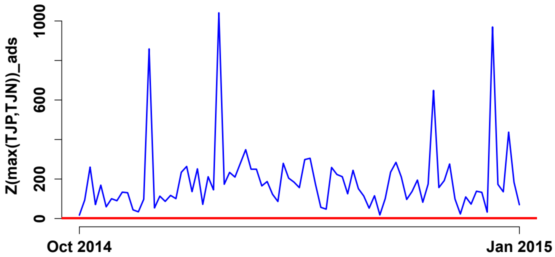


(e) Dyna. Algo on 5 minute

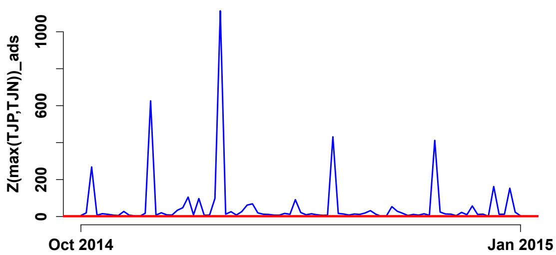


(f) Dyna. Algo on 10 minute

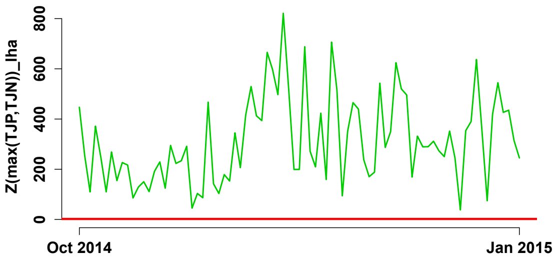
Figure 7: Values of the test statistics of Dynamic algorithm: Dynamic approach for the three stocks of Adidas (blue), Lufthansa (green) and Deutsche Bank (purple) from October 2014 to January 2015 based on 5- and 10-minute respectively.



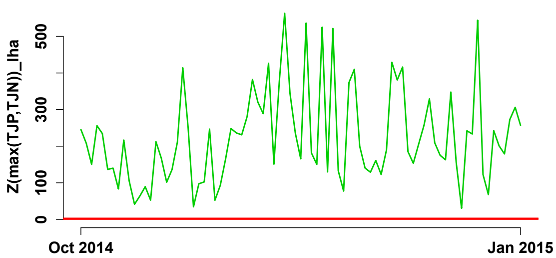
(a) Hal. Algo on 5 minute



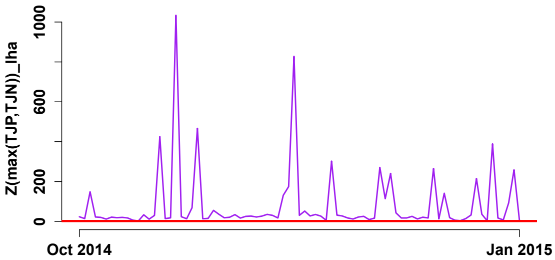
(b) Hal. Algo on 10 minute



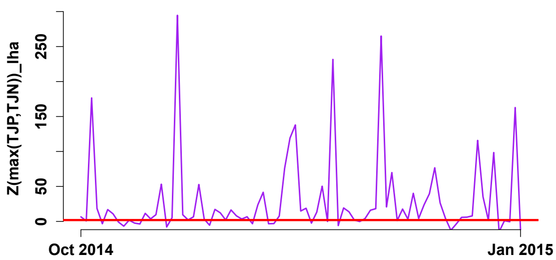
(c) Hal. Algo on 5 minute



(d) Hal. Algo on 10 minute

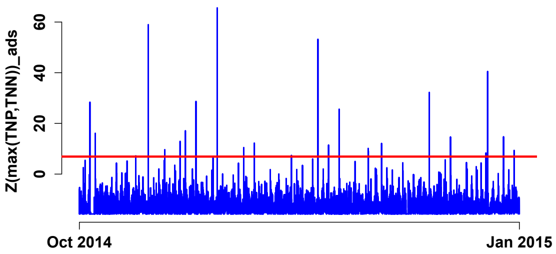


(e) Hal. Algo on 5 minute

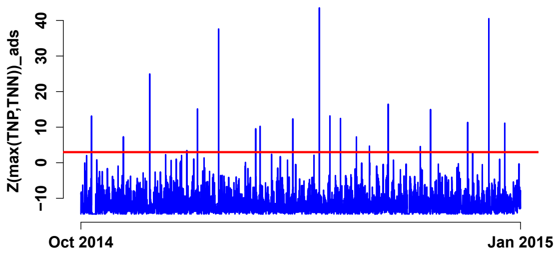


(f) Hal. Algo on 10 minute

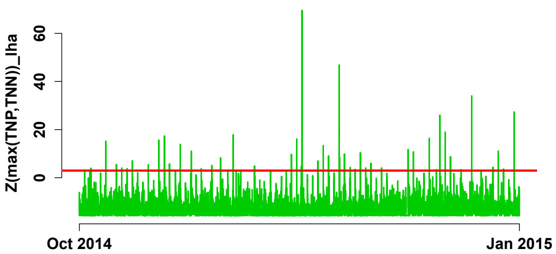
Figure 8: Values of the test statistics of Highs and lows algorithm: Highs and lows approach for the three stocks of Adidas (blue), Lufthansa (green) and Deutsche Bank (purple) from October 2014 to January 2015 based on 5- and 10-minute respectively.



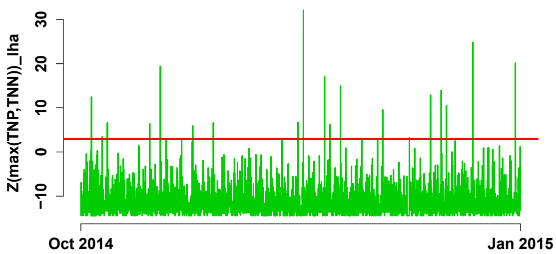
(a) Gumbel Algo on 5 minute



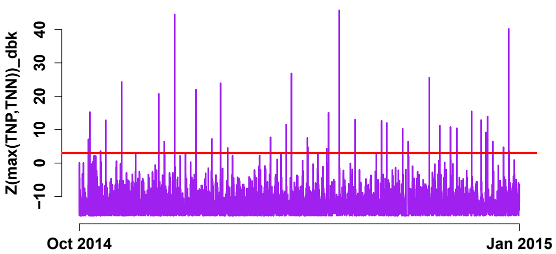
(b) Gumbel Algo on 10 minute



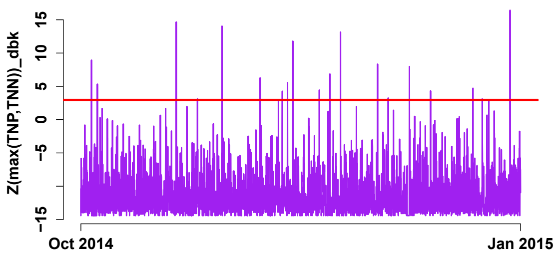
(c) Gumbel Algo on 5 minute



(d) Gumbel Algo on 10 minute



(e) Gumbel Algo on 5 minute



(f) Gumbel Algo on 10 minute

Figure 9: Values of the test statistics of Gumbel test algorithm: Using Gumbel test approach for the three stocks of Adidas (blue), Lufthansa (green) and Deutsche Bank (purple) from October 2014 to January 2015 based on 5- and 10-minute respectively.

can be seen obviously that the number of detected jumps using highs and lows algorithm is considerable compared with other algorithms at each sampling frequency. For lower sampling frequencies of every 5 minutes or 10 minutes, the difference of the number of the detected jumps between the highs and lows algorithm and other algorithms becomes smaller. However, the number of total jumps identified by the highs and lows algorithm is still ten times of that by other two algorithms. The detection rate using the Gumbel test algorithm and the dynamic algorithm is similar at a sampling frequency of every 5 or 10 minutes. For the highest sampling frequency, the dynamic algorithm detects more jumps than the Gumbel test algorithm, where the number of jumps detected by Gumbel test algorithm is roughly one half of the value using the dynamic algorithm. The difference of performance between the Gumbel test algorithm and the other algorithms may be caused by the selection of asymptotic distribution. The test statistic of Gumbel test algorithm is constructed to follow standard Gumbel distribution and the others select standard normal distribution as their asymptotic distribution.

Obviously, the jump intensity at different sampling frequencies using the same algorithm is almost consistent except for the highs and lows algorithm. For all stocks, about 0.6% and 0.5% of jumps are detected using the dynamic algorithm and the Gumbel test algorithm, respectively. On the other side, the jump intensity of the highs and lows algorithm drops at a decreasing rate as the sampling frequency decreases. For instance, when the observations frequency is reduced from every 1 minute to 5 minutes, the jump intensity dramatically drops by 70%. However, it does not change much at a frequency from 5 minutes to 10 minutes. This results shows the highs and lows algorithm overestimates the number of jumps especially at a high frequency. Existence of substantial gradual jumps at a high frequency is likely a reason to the phenomenon. For lower frequencies, mathematical jumps and gradual jumps could cancel out each other, which leads to lower detection rate [?]. In addition, the number of positive jumps is as much as that of negative jumps by all three algorithms at the highest frequency. It means that the positive and negative jumps take place with same probability during the day. However, by the Gumbel test algorithm and the dynamic algorithm the number of positive jumps exceeds negative jumps, which can be interpreted that the robustness to the jumps' sign is shifted to the right. For the lowest data frequency of 10 minutes, it is particularly evident that rare negative jumps are identified by the Gumbel test algorithm, which is probably caused by the selection of thresholds.

At this point, recall that the jump distribution depends on confidence level by Ane and

	Dynamic Algo			Highs and lows Algo			Gumbel test Algo		
	ADS	LHA	DBK	ADS	LHA	DBK	ADS	LHA	DBK
				on 1 *minute*					
No. of jumps									
total	206	123	143	11545	12102	7464	131	137	108
positive	101	70	70	5947	6134	3874	65	75	53
negative	105	53	73	5598	5878	3590	66	62	55
Intensity									
total	0.005	0.003	0.003	0.272	0.285	0.176	0.003	0.003	0.002
positive	0.002	0.002	0.002	0.140	0.145	0.094	0.002	0.002	0.001
negative	0.003	0.001	0.002	0.132	0.139	0.085	0.002	0.002	0.001
				on 5 *minutes*					
No. of jumps									
total	53	74	62	595	575	383	40	49	40
positive	28	41	32	322	314	214	24	31	22
negative	25	33	30	273	261	169	16	18	18
Intensity									
total	0.006	0.009	0.007	0.070	0.067	0.045	0.005	0.006	0.005
positive	0.003	0.005	0.004	0.038	0.037	0.025	0.003	0.004	0.003
negative	0.003	0.004	0.004	0.032	0.031	0.020	0.002	0.002	0.002
				on 10 *minutes*					
No. of jumps									
total	26	24	21	252	199	182	20	21	19
positive	17	15	13	137	109	119	14	15	18
negative	9	9	8	115	90	63	6	6	1
Intensity									
total	0.006	0.006	0.005	0.058	0.046	0.042	0.005	0.005	0.004
positive	0.004	0.004	0.003	0.032	0.025	0.028	0.003	0.004	0.004
negative	0.002	0.002	0.002	0.027	0.021	0.015	0.001	0.001	0.000

Table 2: Number of jumps for asset of Adidas, Lufthansa and Deutsche Bank from October 2014 to January 2015 based on $\alpha = 0.05$ for three data frequency.

	Dynamic Algo			Highs and lows Algo			Gumbel test Algo		
No. of jumps	ADS	LHA	DBK	ADS	LHA	DBK	ADS	LHA	DBK
				$\alpha = 0.05$					
total	53	74	62	595	575	383	40	49	40
positive	28	41	32	322	314	214	24	31	22
negative	25	33	30	273	261	169	16	18	18
				$\alpha = 0.01$					
total	38	60	53	561	543	361	32	35	33
positive	22	34	29	308	295	201	19	23	19
negative	16	26	24	253	248	160	13	12	14
				$\alpha = 0.001$					
total	29	49	44	524	508	338	24	27	28
positive	17	27	25	288	275	190	16	20	17
negative	12	22	19	236	233	148	8	7	11

Table 3: Robustness of amount of detected jumps for $\alpha = \{0.05,\ 0.01,\ 0.001\}$ based on 5-minute data frequency. Confidence level is equal to $1 - \alpha$.

Metais [?]. Consequently, the number of detected jumps differs with the selected confidence levels. Thus, it is important to test the robustness of the three algorithms for the jump detection to the selection of confidence levels with 0.95, 0.99 and 0.999. Intuitively, when the value of confidence level is lowered, the confidence interval is narrowed accordingly and the overall number of jumps increases. The empirical evidence in Table **??** reveals that the highs and lows algorithm robustly estimates the number of jumps, including the positive and the negative jumps. For instance, when $\alpha = 0.05$ the total number of jumps for the stock Adidas by the highs and lows algorithm rises to 595, and when $\alpha = 0.001$ the number of jumps remains equal to 524. In comparison, the number of detected jumps has an overall downward trend as the confidence level increases (α decreases) using the dynamic algorithm and the Gumbel test algorithm. Indeed, the total number of jumps drags a cumulative drop of 50% at a confidence level from 0.95 to 0.999 by these two algorithms. As expected, for each stock index there is no significant statistical difference in the detection rate using the same algorithm. With respect to each confidence level, more positive jumps are identified than negative ones, as expected. To conclude, both of the dynamic algorithm and the Gumbel test algorithm are more sensitive to the selection of confidence levels than the highs and lows algorithm, which means that the estimates by these two algorithms are not always reliable. On the other hand, the analysis of results by the highs and lows algorithm from Table **??** highlights the robustness of the jump detection technique of this algorithm.

4.3 Some statistics of detected jumps

As introduced by Lee and Mykland [**?**], the jump arrival timing is not regular, so that the constant jump intensity is not appropriate to explain the jump arriving timing. From overall other empirical results one can expect that most jumps in asset price process arrive around the time of a market's opening and closing. In addition, the jumps are associated with the releases of important news, such as the announcement of the third-quarter earnings. The reason that the majority of jumps occur during the first hour of the trading time can be explained by "the pressure at a market opening after a long period of interrupted trading and the accompanying accumulated information", which is proposed by Ane and Metais [**?**]. Then the analysis of jump arrival timing is proceeded for further empirical study. In Figures **??**, **??** and **??**, the performance of three DAX stock return process from October 2014 to January 2015 at a frequency of 5 minutes are presented, respectively. In part (a), (c) and (e) of the above three figures illustrate the comparison of three discussed algorithms that indicate the proportion between positive jumps and negative jumps. In (b), (d) and (f) sub-figures put the focus on the exact jump arrival timing during one day over the period of four months.

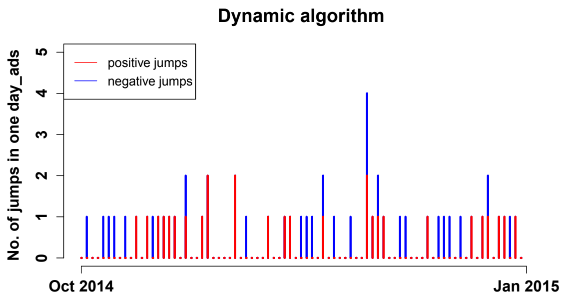
From an overall perspective of (a), (c) and (e) sub-figures, the number of the total jumps is different using the three different algorithms for each stock. Moreover, the proportion of positive jumps is relatively higher by each algorithm. As shown in the part (c) of three corresponding figures, both of the positive and the negative jumps can be identified almost every day. In contrast, only few days with either a positive or negative jump are observed by the dynamic algorithm and the Gumbel test algorithm. In the part of (b), (d) and (f) of Figures **??**, **??** and **??**, the specific situations of intradaily jump arrival timing for each stock are clearly visualized. The position of blue point represents the total number of jumps arriving at this time point and the green and red point show the value of positive jumps and negative jumps, respectively. For the same stock, the performance of the dynamic algorithm is similar as the Gumbel test algorithm. By the same algorithm, all three stocks share the similar pattern of jump arrival timing in one day. Most jumps are recorded in the first and the last hour of the trading time, which coincides with the understanding of a jump as a

	Max	Min	Mean	St.Dev	>= 15	<= 3
			Adidas			
Dynamic Algo	4	0	0.654	0.727	0	80
Highs and lows Algo	23	0	7.169	4.531	6	19
Gumbel test Algo	2	0	0.289	0.482	0	83
			Lufthansa			
Dynamic Algo	4	0	0.914	0.809	0	80
Highs and lows Algo	27	1	6.928	4.785	7	19
Gumbel test Algo	3	0	0.590	0.606	0	83
			Deutsche Bank			
Dynamic Algo	3	0	0.765	0.841	0	81
Highs and lows Algo	15	0	4.615	2.663	1	33
Gumbel test Algo	2	0	0.482	0.571	0	83

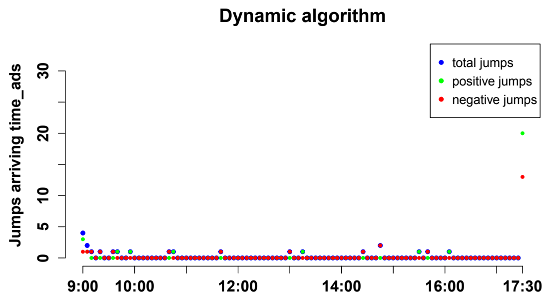
Table 4: The statistics of number of detected jumps in one day, with different algorithms and stocks based on 5-minute frequency.Max, Min, Mean and St.Dev stand for the maximum, minimum and average number of jumps and standard deviation of jumps per day respectively.>= 15 and <= 3 represent the number of days with more than 15 and less than 3 jumps respectively.

reaction of additional information within a diffusion process. All three algorithms reveal this pattern apparently.

As shown in Table **??**, some basic statistics of daily jumps are reported by different algorithms and stocks. The maximum, minimum and average number of daily jumps are considered to indicate the extreme cases of jump occurrence. Additionally, the standard deviation of number of daily jumps and the days with more than 15 jumps and less than 3 jumps are also summarizes in this table. As expected, the maximum number of daily jumps detected by the highs and lows algorithm is much larger than that of the other algorithms. The maximum number of the dynamic algorithm slightly exceeds the number of the Gumbel test algorithm. For the stock of Lufthansa, the performances of the Gumbel test algorithm and the highs and lows algorithm show the biggest difference. By the highs and lows algorithm, the maximum number of daily jumps reaches up to 27, while it is only 3 jumps by the Gumbel test algorithm. However, the three algorithms for each stock show roughly same



(a)



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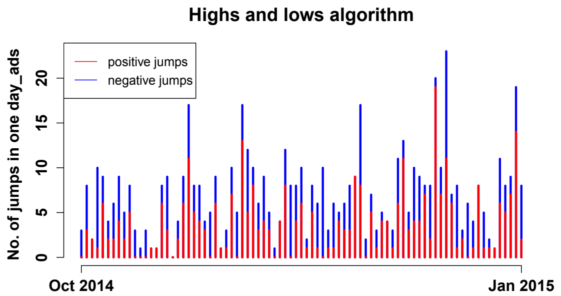

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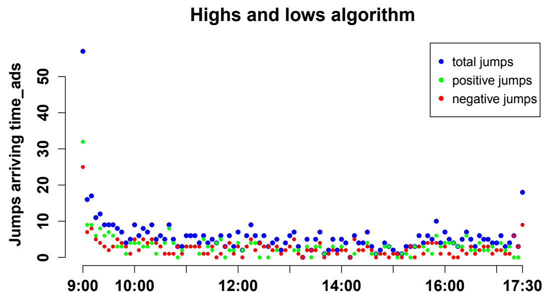
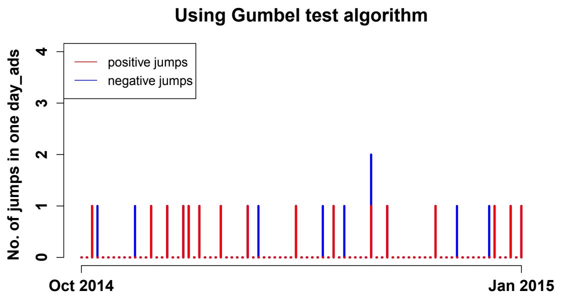
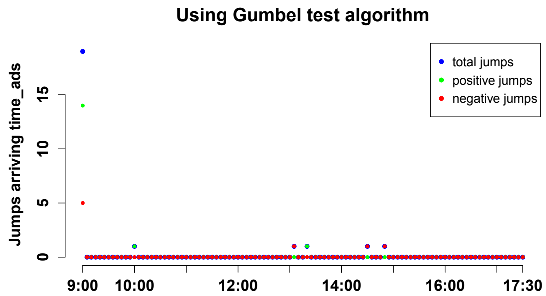


(d)

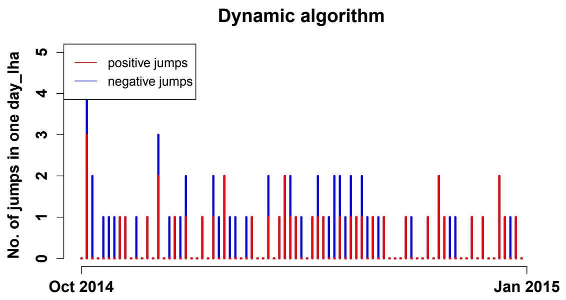


(e)

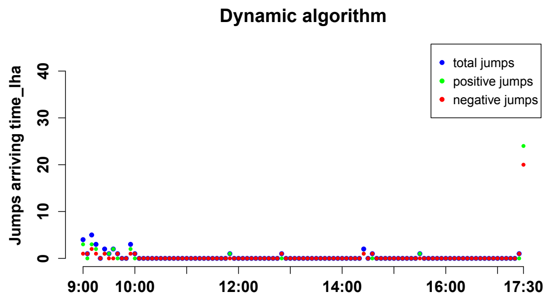


(f)

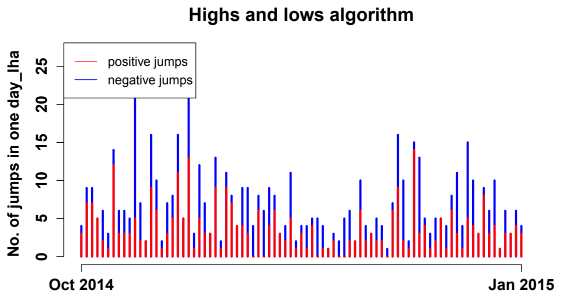
Figure 10: Number of positive, negative and total jumps within one day (left) and number of jumps according to arriving time (right) for stock: Adidas based on 5-minute interval. (a),(c) and (e) sub-figures present the proportion between positive jumps and negative jumps of Adidas. (b),(d) and (f) present the exact jump arrival timing of Adidas.



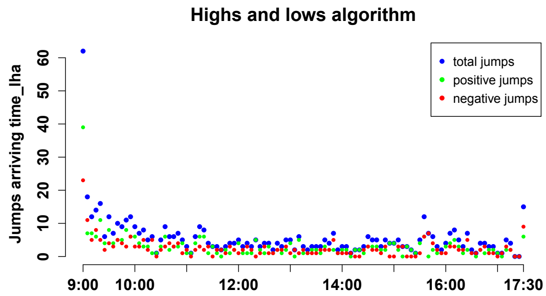
(a)



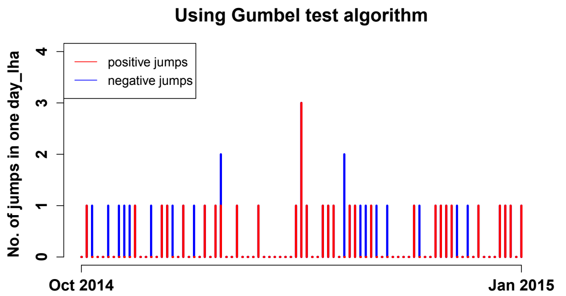
(b)



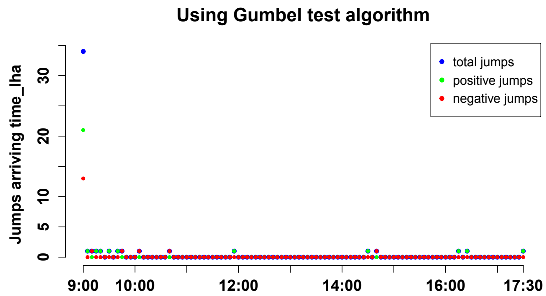
(c)



(d)



(e)



(f)

Figure 11: Number of positive, negative and total jumps within one day (left) and number of jumps according to arriving time (right) for stock: Lufthansa based on 5-minute interval. (a),(c) and (e) sub-figures present the proportion between positive jumps and negative jumps of Lufthansa. (b),(d) and (f) present the exact jump arrival timing of Lufthansa.

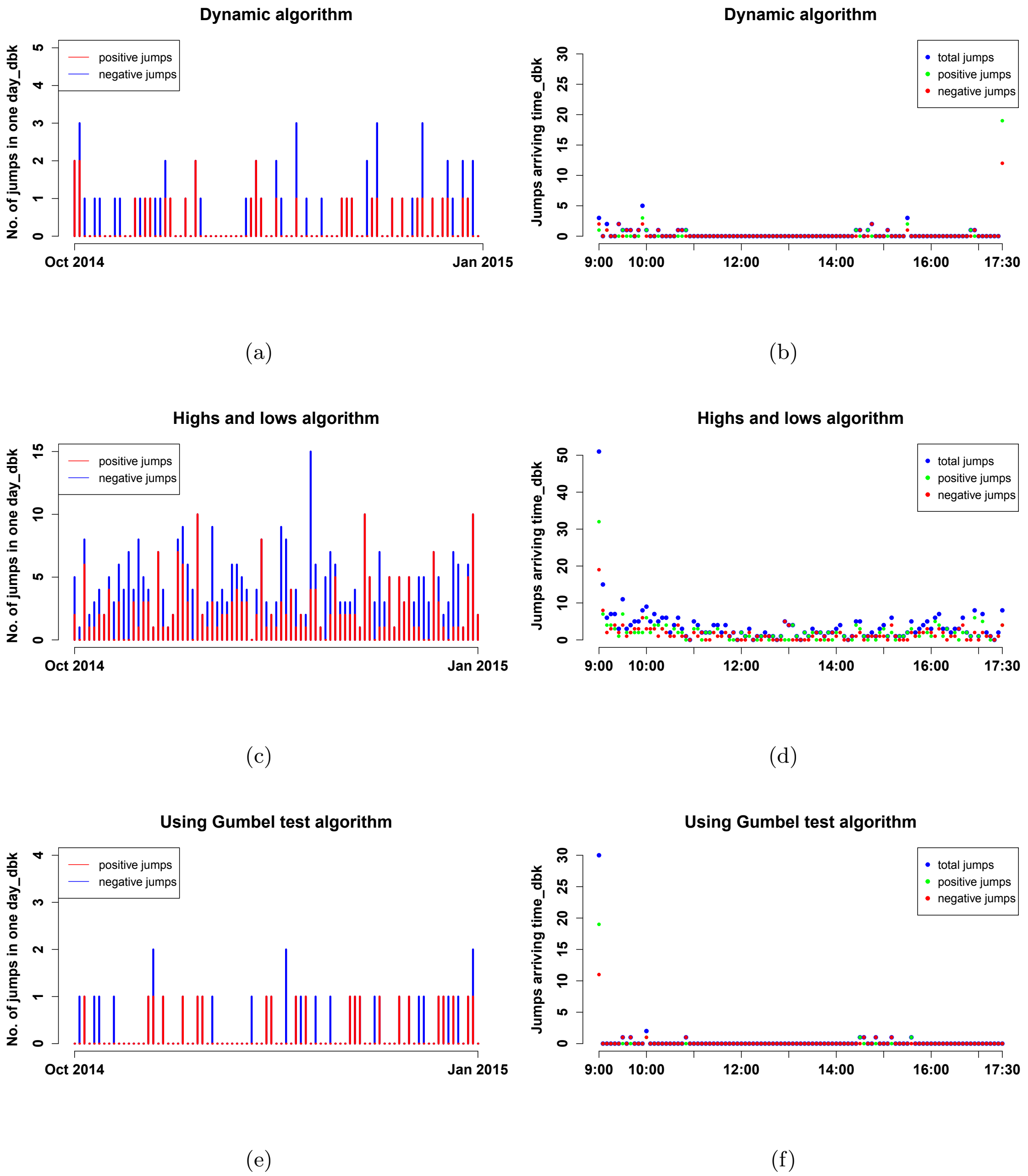


Figure 12: Number of positive, negative and total jumps within one day (left) and number of jumps according to arriving time (right) for stock: Deutsche Bank based on 5-minute interval. (a),(c) and (e) sub-figures present the proportion between positive jumps and negative jumps of Deutsche Bank. (b),(d) and (f) present the exact jump arrival timing of Deutsche Bank.

	Number of jumps in first hour			Number of jumps in last hour		
	total	positive	negative	total	positive	negative
			Adidas			
Dynamic Algo	11	6	5	33	20	13
Highs and lows Algo	164	94	70	72	35	37
Gumbel test Algo	32	19	13	0	0	0
			Lufthansa			
Dynamic Algo	22	14	8	45	24	21
Highs and lows Algo	189	110	79	43	19	24
Gumbel test Algo	41	26	15	1	1	0
			Deutsche Bank			
Dynamic Algo	16	6	10	33	20	13
Highs and lows Algo	125	68	57	44	25	19
Gumbel test Algo	32	19	13	0	0	0

Table 5: Recorded jumps time in the first and last hour of the trading time reported by Dynamic Algo, Highs and lows Algo and Gumbel test Algo for each stock based on 5-minute data frequency.

result for the minimum number of daily jumps with all around zero. Only for Lufthansa, at least one jump is identified in each day by the highs and lows algorithm.

For average number of daily jumps, three underlying stocks of Adidas, Lufthansa and Deutsche Bank are all detected the most jumps by the highs and lows algorithm, 7.169, 6.928 and 4.615 jumps every day, respectively. Overall, the average daily jumps detected by the highs and lows algorithm are almost 5 and 10 times as much as those by the dynamic algorithm and the Gumbel test algorithm, respectively. On the other hand, a large standard deviation indicates a striking difference among the numbers of detected daily jumps. Similar as average number of daily jumps, the biggest standard deviation of number of daily jumps is also identified using the highs and lows algorithm with the value of around 4.531 and 4.785 for the stock of Adidas and Lufthansa and 2.663 for Deutsche Bank.

For further studies, the number of days with more than fifteen jumps and less than three jumps are sorted to analyze the some extreme situations. By the dynamic algorithm, as high as 80 days on average with less than three jumps are reported for each stock. Compared with the dynamic algorithm, the situation of the Gumbel test algorithm is more extreme, with identifying all days with less than three jumps. To find out the number of days with more than fifteen jumps, the highs and lows algorithm reports the highest value on this number as expected. The most surprising finding of this study is that no days are recorded with more than fifteen jumps by the dynamic algorithm and the Gumbel test algorithm. Furthermore, up to 7 days with more than fifteen jump are identified for Lufthansa, and 6 for Adidas. For the stock of Deutsche Bank, it seems that nearly few days are reported with extreme high value of daily jumps, even if the highs and lows algorithm is applied.

Table **??** summarizes the counts of jump occurrences during the first and last hour of the trading time. Comparing all the algorithms, the largest numbers of jump occurrences both in the first and last hour are observed by the highs and lows algorithm. More jumps detected by the dynamic algorithm appear in the last hour than by the Gumbel test algorithm. However, the majority of jumps detected by the Gumbel test algorithm gathers in the first hour. Generally, more positive jumps are detected than negative ones in the first and last hour regardless of algorithm selection. The number of positive jumps for Deutsche Bank exceeds that of negative ones using the highs and lows algorithm and the Gumbel test algorithm, however, while the result of dynamic algorithm is in opposite. Only few jumps are obtained by the Gumbel test algorithm, especially in the last hour with almost zero, which yields the fewest jumps detected as shown previously.

4.4 Jump size

After the study of the jump occurrence during one day, the jump size is the next problem to be investigated. Table **??** summarizes the typical descriptive statistics for positive and negative jump sizes at a frequency of 5-minute measured by the three discussed algorithms. Obviously, the results differ so much between the highs and lows algorithm and the other algorithms. The jump size of positive jumps and negative jumps by the Gumbel test algorithm is as large as that by the dynamic algorithm, whose jump size is almost 100 times by the highs and lows algorithm. A reasonable explanation is that except for the highs and lows algorithm, the jump size of the other algorithms is calculated directly by the change of log-returns on days with a significant jump test statistic. Meanwhile, jump size by the highs and lows algorithm is obtained by the estimate of SSJ. Associated with the existence of overestimation of jumps, thus the jump size by the highs and lows algorithm is quite small accordingly. In addition, the asymmetric jump sizes can be seen from the table. The positive jump size is a little bigger than the negative one by each algorithm, which indicates an stronger reaction to the release of news that causes larger positive jumps. Finally, the skewness and excess kurtosis of the jump size are shown totally different from zero for all three stocks, which is against the assumption of standard normal distribution.

	Mean	Standard deviation	Skewness	Excess Kurtosis
		positive jump size		
Dynamic Algo				
ADS	1.227×10^{-2}	1.119×10^{-2}	2.256	3.947
LHA	1.094×10^{-2}	7.155×10^{-3}	2.421	8.156
DBK	1.110×10^{-2}	6.301×10^{-3}	0.842	0.235
Highs and lows Algo				
ADS	1.059×10^{-4}	8.840×10^{-4}	12.146	153.110
LHA	9.065×10^{-5}	3.992×10^{-4}	11.404	157.774
DBK	9.766×10^{-5}	3.481×10^{-4}	6.338	47.831
Gumbel test Algo				
ADS	1.368×10^{-2}	1.214×10^{-2}	1.895	2.290
LHA	1.220×10^{-2}	7.390×10^{-3}	2.547	7.856
DBK	1.338×10^{-2}	5.924×10^{-3}	0.851	0.149
		negative jump size		
Dynamic Algo				
ADS	-8.144×10^{-3}	4.637×10^{-3}	-2.491	6.583
LHA	-9.596×10^{-3}	4.535×10^{-3}	-1.019	0.566
DBK	-9.318×10^{-3}	4.507×10^{-3}	-2.401	6.769
Highs and lows Algo				
ADS	-2.922×10^{-5}	1.821×10^{-4}	-15.448	246.052
LHA	-4.757×10^{-5}	1.075×10^{-4}	-5.380	32.724
DBK	-5.487×10^{-5}	1.640×10^{-4}	-5.995	40.806
Gumbel test Algo				
ADS	-9.382×10^{-3}	5.329×10^{-3}	-1.998	3.351
LHA	-1.188×10^{-2}	4.503×10^{-3}	-0.886	-0.434
DBK	-1.068×10^{-2}	5.385×10^{-3}	-1.801	2.986

Table 6: Descriptive statistics for positive and negative jump sizes for the stocks of Adidas, Lufthansa and Deutsche Bank based on $\alpha = 0.05$ for 5-minute data frequency.

5 Monte Carlo simulation

In this section, the effectiveness of all algorithms are evaluated by a Monte Carlo simulation study. As earlier mentioned, in order to obtain consistent estimators, the sampling interval is required to converge to zero asymptotically. However, this requirement can not be achieved in the real applications. On the other side, the stock return process has been proved from the empirical study that it is non-Gaussian distributed, but owns a feature of heavy tails and high peaks followed by Andersen et.al [?]. The leverage effect is also important in the empirical work about stock returns when a negative return sequence is associated with increases in volatility. Therefore, in this simulation, Heston stochastic volatility model [?] is applied whose setting takes account of non-lognormal distribution of the assets returns, leverage effect, and an important mean-reverting property of volatility.

5.1 Process modeling

Then a price process can be generated by the Heston stochastic volatility model, which is very close to the observed price process in real applications. The Heston stochastic volatility model is described by the bi-variate stochastic process for the equity stock price S_t and its variance V_t

$$dS_t = rS_t dt + \sqrt{V_t} S_t dW_t^1 + J_t dP_t, \tag{53}$$

$$dV_t = \kappa(\theta - V_t)dt + \sigma\sqrt{V_t} dW_t^2, \tag{54}$$

$$dW_t^1 dW_t^2 = \rho dt, \tag{55}$$

where r denotes the rate of the equity return. The jump component $J_t dP_t$ can be decomposed into its counting process P_t and the jump size J_t. W_t^1 and W_t^2 are Standard Brownian movements. Considering the leverage effect, $\{W_t^1\}_{t\geq 0}$ and $\{W_t^2\}_{t\geq 0}$ are correlated with the parameter $\rho \neq 0$. The Equation **??** is captured for the variance which is known as the square-root mean reverting process of Feller [?] and Cox, Ingersoll and Ross [?]. θ and κ are average price variance in long run and the rate of reversion, while σ is referred to the variance of $\{V_t\}_{t\geq 0}$. As proposed by Mikhailov and Noegel [?], the variance of square root process is always positive and if $2\kappa\theta > \sigma^2$ then it cannot reach zero. Moreover, the authors emphasized that the deterministic part of process in Equation **??** is asymptotically stable when $\kappa > \theta$.

5.2 Simulation procedure

In order to calculate the probabilities of spurious detection jump and success in detecting an actual jump, respectively, the simulations are conducted in two cases: with and without jumps. The data requirement in simulation is different by each algorithm. For the dynamic algorithm, one thousand series of spot prices over three months are simulated at several data frequencies from 5-minute to 30-minute returns. For the highs and lows algorithm, 1000 simulated series of a log-price process in one day are implemented under the same data frequencies, respectively. For the Gumbel test algorithm, 1000 series of return process are simulated with the grid-size $n = 65$, 120, and 160 (similar sample size as before) to construct the empirical distribution function. When incorporated with jumps, the number was set to be one in each series. The jump arrival time is selected randomly from a uniform distribution. The significance level for this study is 5%. Three different simulation models are applied to generate spot price series. Then compare the effectiveness of test statistics using the generated spot price process. The simplest simulation model is with a fixed volatility, while the others are both for stochastic volatility. The main difference of two stochastic volatility models is that one assumes no correlation between the Brownian motion in volatility and the random terms in the return process and the other one is considered with a leverage effect.

The model with a constant volatility sets σ directly at 30% and 60% of the value of V_0. The stochastic volatility model is specified by discretising the stochastic process using Euler discretization scheme. Start with the initial values S_0 for the stock price and V_0 for the variance. Given a value for V_t at time t, V_{t+dt} can be obtained from

$$V_{t+dt} = V_t + \kappa \cdot (\theta - V_t)dt + \sigma\sqrt{V_t dt} Z_v. \tag{56}$$

and S_{t+dt} is obtained from

$$S_{t+dt} = S_t + r \cdot S_t dt + \sqrt{V_t dt} S_t Z_s + S_t \cdot J_t. \tag{57}$$

If there is no leverage effect in the simulation, Z_v is independent from Z_s. If the leverage effect is taken into account, two independent standard normal variables Z_1 and Z_2 are first generated to obtain Z_v and Z_s with correlation ρ. Set $Z_v = Z_1$ and $Z_s = \rho Z_1 + \sqrt{1-\rho^2} Z_2$.

5.3 Simulation results

Table **??** presents the assumed model parameters for the simulation study. The Monte Carlo simulation was performed under three corresponding algorithms based on several data frequencies. Furthermore, the results are compared with each other using different discretization

Variable	Value	Variable	Value
S_0	100	V_0	0.010
r	0	σ_0	0.010
θ	0.010	σ_1	0.003
κ	2	σ_2	0.006
ρ_1	-0.620	ρ_2	0.620

Table 7: Designed model parameters for the simulation study

schemes, including with a constant volatility, stochastic volatility without leverage effect and stochastic volatility with leverage effect.

Table **??** summarizes the probability of spurious detection of jumps. To see the effect of the jumps on the detection rate of the test statistics, three discussed algorithms of jump detection were applied again according to three data frequencies. As mentioned before, here spot price processes were simulated by three simulation models, one with a fixed volatility and the others with stochastic volatility. 1000 series of return process without jumps were simulated to calculate the spurious detection rate of jumps. Comparing to other two algorithms, the dynamic algorithm performed best with the smallest spurious detection rate by almost zero whatever data frequency was chosen. In contrast, the highs and lows algorithm performed worst. Based on 1000 simulated days, the result shows more than 90% of the days were detected by one jump although it was set no jumps in the simulation model. This supports the suspicion that the overestimation of jumps exists using the highs and lows algorithm. On the other hand, the spurious detection rate of jumps using the Gumbel test algorithm was close to $10\% - 20\%$, much higher than the dynamic algorithm. With respect to the different simulation models, the effects of all these models using the same algorithm were similar. A more complex model did not have better performance for the problem of spurious detection than a simplest model. Except for the Gumbel test algorithm, the simulation results of the other algorithms suggested that the test statistics are robust in spurious detection of jumps for sampling frequencies ranging from 5 to 30 minutes. However, under the Gumbel test algorithm the probability of spurious detection of jumps was reduced as the sampling frequency increased.

Next, one thousand tests at different sampling frequencies from 5 to 30 minutes were performed to detect actual jump. Table **??** shows the probability of success in detecting an

	σ_1	σ_2	SV	ρ_1	ρ_2
			5-minute returns		
Dynamic Algo	0.001	0.002	0.000	0.002	0.000
Highs and lows Algo	0.998	0.999	0.994	0.998	0.995
Gumbel test Algo	0.091	0.122	0.095	0.112	0.101
			10-minute returns		
Dynamic Algo	0.004	0.001	0.000	0.000	0.001
Highs and lows Algo	0.974	0.981	0.925	0.909	0.906
Gumbel test Algo	0.083	0.143	0.159	0.103	0.094
			30-minute returns		
Dynamic Algo	0.003	0.000	0.005	0.001	0.000
Highs and lows Algo	0.976	0.982	0.959	0.941	0.952
Gumbel test Algo	0.201	0.226	0.197	0.145	0.129

Table 8: Spurious detection rates of jump are calculated on every 5, 10- and 30-minute returns, respectively. Constant volatility sets $\sigma = \sigma_1$ or σ_2. ρ_1 and ρ_2 present stochastic volatility model with leverage effect, assuming $\rho = \rho_1$ or ρ_2. The significance level α is 5%.

actual jump. In each diffusion process only one jump was incorporated. The simulation was taken based on the Heston stochastic volatility model which considers leverage effect with $\rho = -0.62$. In order to evaluate the difficulty of the test statistics for a success in jump detection, different jump sizes have to be configured. Six different levels of jump sizes are assumed at 10-200% of the underlying volatility level. The jump arriving time is set to be the same as in the case of calculation of spurious detection rate.

As is shown in Table **??**, very high detection power of a actual jump with more than 96% was obtained at the each frequency when Gumbel test algorithm was applied, even for very small-sized jumps. For the dynamic algorithm, the detection rate was falling rapidly as the relevant jump size was reduced. As argued by Lee and Mykland [**?**], the price changes resulting from diffusion are less likely to be distinguished from those from the actual jump especially at a lower sampling frequency. For instance, only 0.8% of jumps could be detected when the jump size was equal to 25% of the volatility at a frequency of every 30 minutes. Even for a higher sampling frequency of every 5 minutes, the detection power was still poor, only with more than 5.3%, which was much smaller than that of the Gumbel test algorithm.

	2σ	1σ	0.5σ	0.25σ	0.2σ	0.1σ
			5-minute returns			
Dynamic Algo	1.000	0.999	0.987	0.237	0.065	0.001
Highs and lows Algo	1.000	1.000	0.998	0.999	0.999	0.954
Gumbel test Algo	0.997	1.000	0.993	0.998	0.992	0.987
			10-minute returns			
Dynamic Algo	0.999	0.889	0.827	0.053	0.008	0.001
Highs and lows Algo	1.000	1.000	0.999	0.991	0.974	0.881
Gumbel test Algo	0.997	0.999	0.985	0.997	0.985	0.981
			30-minute returns			
Dynamic Algo	0.998	0.979	0.158	0.008	0.002	0.002
Highs and lows Algo	0.993	0.994	0.993	0.967	0.862	0.706
Gumbel test Algo	0.999	0.986	0.985	0.974	0.962	0.983

Table 9: Probability of success in detecting of actual jumps based on 5, 10 and 30 minutes returns. The encompassing model is Heston stochastic volatility model with $\rho = -0.62$. The significance level α is 5%. The jump sizes are set in comparison with volatility level: 2σ means the jump sizes are set three times of V_{t-1}.

Although the one jump was identified with a high precision using highs and lows algorithm, the spurious detection rate of this algorithm was unfortunately still very large in the same way. Therefore, it is difficult to tell whether there is an actual jump within the diffusion process. As already suspected, highs and lows algorithm is likely to overestimate the number of jumps significantly on a given day. On the other hand, this algorithm is able to detect the smallest-sized jumps even at a low frequency compared with the dynamic algorithm. More than 70% of smallest-sized jumps were identified correctly using the highs and lows algorithm, while the dynamic algorithm detected only 0.2% of jumps at the same data frequency of 30-minute returns.

6 Conclusions

In this thesis, three different algorithms were introduced to detect jumps in a continuous jump-diffusion process. The first one is called dynamic algorithm developed by Lee and Mykland [?]. The dynamic algorithm can dynamically identify jumps and determine the exact jump arrival timing simultaneously. The second one, so-called the highs and lows algorithm, is based on the theory developed by Kloessner [?]. The idea of using intradaily highest and lowest returns during a sub-period to construct the test statistic was first presented by Andersen et.al[?]. This algorithm includes the data information as more as possible and leads to the problem of overestimation, especially at a ultra high sampling frequency. Positive and negative jumps can be separately identified by using two different test statistics. The third algorithm, called the Gumbel test algorithm, was introduced by Palmes and Woerner [?]. Based on the idea of extreme value theory, the maximum of increments of a Brownian semi-martingale process follows Gumbel distribution, in the absence of jumps. In order to detect negative jumps, the original return process is set to minus to find out the maximums of the new process for further jump detection.

All three algorithms were applied to a database containing three stock spot prices of Adidas, Lufthansa and Deutsche Bank from October 2014 till January 2015. Data entries outside the trading time were deleted for the purpose of completeness. During this period, European debt crisis was going on, resulting in existence of jumps for specific days.

The empirical results showed that the number of jumps detected by the highs and lows algorithm was much larger than that of other two algorithms. The main difference was due to the influence of gradual jumps. The highs and lows algorithm is uniquely able to detect gradual jumps. Moreover, the problem of overestimation for the jump detection at a ultra high frequency led a significant test statistic of this algorithm. For the highest frequency of 1 minute, total jumps detected by the dynamic algorithm were twice as those detected by the Gumbel test algorithm. However, for a lower sampling frequency, there was essentially no statistical difference in the number of jumps between the dynamic algorithm and the Gumbel test algorithm.

On the other side, jump detection rate differed to confidence levels for each algorithm. Based on the empirical results, the dynamic algorithm and the Gumbel test algorithm were more sensitive to the selection of confidence levels than the highs and lows algorithm. With the confidence level from 0.95 to 0.999, the detection rate of the dynamic algorithm and the Gumbel test algorithm dropped by about 40%, while the value using the highs and lows algorithm was only reduced by 10%. The results of the highs and lows algorithm seemed to be robust to the selected confidence level, while the results of the dynamic algorithm and the Gumbel test algorithm were not stable at the highest sampling frequency. Thus, the data can not be chosen too high, when the dynamic algorithm and the Gumbel test algorithm are used.

The findings of jump arrival timing showed that most jumps occurred in the first and the last hour of the trading time. In accordance with the results of total number of jumps, the highs and lows algorithm detected the most jumps both in the first and last hour. In the simulation, the majority of jumps detected by the Gumbel test algorithm gathered in the first hour, while most jumps were detected in the last hour using the dynamic algorithm.

Furthermore, the highs and lows algorithm reported some extreme high value of daily jumps, such as 7 days with more than fifteen jumps were identified for Lufthansa, and 6 days for Adidas. In comparison with the highs and lows algorithm, almost all days were reported with less than three jumps by the dynamic algorithm and the Gumbel test algorithm.

For all three algorithms, the positive jump size was a little bigger than the negative one. And the distribution of the jump size was demonstrated against the assumption of standard normal distribution.

In order to verify the effectiveness of the corresponding test statistic, it is necessary to conduct a Monte Carlo simulation. One thousand series of spot prices over three months were simulated at several data frequencies from 5-minute to 30-minute for each algorithm. The spurious detection rate and the probability of success in detecting an actual jump were both calculated to evaluate the performance of each algorithm. The simulation showed the following important results. Firstly, the dynamic algorithm performed best on spurious detection rate. But for detection power for a small-sized jump, it is unable to detect jumps with high precision. Secondly, the highs and lows algorithm suffered the problem of overestimation for jump detection. Thus, the spurious detection rate was unexpectedly high using this algorithm. Finally, the Gumbel test algorithm had a superior performance in calculating the probability of success in detecting of actual jumps. Even small-sized jumps of 10% of volatility were detected with more than 98% accuracy.

This thesis focused on detecting the Poisson jumps in simulation. For further research, other kinds of jumps can be incorporated within the stochastic process.

References

[1] S. Lee and P. Mykland, "Jumps in financial markets: A new nonparametric test and jump dynamics," *Review of Financial Studies*, vol. 21, pp. 2535–2563, 2008.

[2] S. Kloessner, "Grasping economic jumps by sparse sampling using intradaily highs and lows." Saarland University, Tech. Rep., 2010.

[3] C.Palmes and J. Woerner, "The gumbel test for jumps in stochastic volatility models," Dortmund Technical University, Tech. Rep., 2013.

[4] S. Heston and S. Nandi, "A close-from garch option valuation model," *The Review of Financial Studies*, vol. 13, pp. 585–625, 2000.

[5] R. Merton, "Option pricing when the underlying stock returns are discontinuous," *Journal of Financial Economics*, vol. 3, pp. 125–144, 1976a.

[6] T. Ane and C. Metais, "Jump distribution characteristics: Evidence from european stock markets." *International Journal of Business and Economics*, vol. 9, no. 1, pp. 1–22, 2010.

[7] T. G. Andersen, T. Bollerslev, P. Frederiksen, and M. Nielsen, "Continuoustime models, realized volatilities, and testable distributional implications for daily stock returns," University of Aarhus, Tech. Rep., 2007b.

[8] O. E. Barndorff-Nielsen and N. Shephard, "Power and bipower variation with stochastic volatility and jumps." *Journal of Financial Econometrics*, vol. 2, pp. 1–37, 2004.

[9] ——, "Econometrics of testing for jumps in financial economics using bipower variation." *Journal of Financial Econometrics*, vol. 4, pp. 1–30, 2006.

[10] L. Zhang, P. Myland, and Y. Ait-Sahalia, "A tale of two time scales: Determing integrated volatility with noisy high-frequency data." *Journal of the American Statistical Association*, vol. 100, pp. 1394–1411, 2005.

[11] J. Hasbrouck, "Empirical market microstructure: Economic and statistical perspectives on the dynamics of trade in securities markets." Stern School of Business, New York University, Tech. Rep., 2005.

[12] P. Hansen and A. Lunde, "Realized variance and market microstructure noise." *Journal of Business & Economic Statistics*, vol. 24, no. 2, pp. 127–161, 2006.

[13] F. Bandi and J. Russel, “Separating microstructure noise from volatility,” *Journal of Financial Economics*, vol. 79, pp. 655–695, 2006.

[14] J. Navas, “Calculation of volatility in a jump-diffusion model,” The Journal of Derivatives, Tech. Rep. G13, 2011.

[15] A. Denmark, “Estimating quadratic variation using realized variance,” *Journal of applied econometrics*, vol. 17, pp. 457–477, 2002.

[16] O. E. Barndorff-Nielsen and N. Shephard, “Econometric analysis of realized volatility and its use in estimating stochastic volatility models.” *Journal of the Royal Statistical Society*, vol. 64, no. 2, pp. 253–280, 2002.

[17] M. Podolskij and M. Vetter, “Estimation of volatility functionals in the simultaneous: Presence of microstructure noise and jumps.” Ruhr university of Bochum, Tech. Rep., 2006.

[18] T. G. Andersen, T. Bollerslev, F. Diebold, and P. Labys, “Modeling and forecasting realized volatility,” *Econometrica*, vol. 71, pp. 579–625, 2003.

[19] G. J. Jiang and R. Oomen, “A new test for jumps in asset prices. working paper, eller college of management,” Eller College of Management, University of Arizona, Tech. Rep., 2005.

[20] S. Kloessner, “Separting risk due to diffusion, positive jumps, and negative jumps.” Saarland University, working paper, 2009.

[21] K. Christensen and M. Podolskij, “Range-based estimation of quadratic variation.” Aarhus School of Business, Tech. Rep., 2006.

[22] O. E. Barndorff-Nielsen, P. R. Hansen, A. Lunde, and N. Shephard, “Realized kernels in practice: Trades and quotes.” *Econometrics Journal*, vol. 4, pp. 1509–1511, 2008b.

[23] J. X. Xiong and T. M. Idzorek, “The impact of skewness and fat tails on the asset allocation decision,” *Financial Analysts Journal*, vol. 67, no. 2, pp. 23–35, 3-4 2011.

[24] T. Andersen, L. Benzoni, and J. Lund, “An empirical investigation of continuous-time equity return models.” *Journal of Finance*, vol. 57, pp. 1239–1284, 2002.

[25] S. Heston, “A closed-form solutions for options with stochastic volatility,” *Review of Financial Studies*, vol. 6, pp. 327–343, 1993.

[26] W. Feller, “Two singular diffusion problems,” *Annals of Mathematics*, vol. 54, pp. 173–182, 1951.

[27] J.Cox, J. Ingersoll, and S. Ross, “A theory of the term structure of interest rates.” *Econometrica*, vol. 53, pp. 389–408, 1985.

[28] S. Mikhailov and U. Noegel, “Hestons stochastic volatility: Model implementation, calibration and some extensions.” *Wilmott*, vol. July, pp. 74–79, 2003.

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